arXiv:math/0105258v2 [math.PR] 8 Jan 2002

Multi-scale homogenization with bounded ratios and Anomalous Slow Diffusion.

Gérard Ben Arous*and Houman Owhadi†

July 30, 2013

Abstract

We show that the effective diffusivity matrix $D(V^n)$ for the heat operator $\partial_t-(\Delta/2-\nabla V^n\nabla)$ in a periodic potential $V^n=\sum_{k=0}^n U_k(x/R_k)$ obtained as a superposition of Holder-continuous periodic potentials U_k (of period $\mathbb{T}^d:=\mathbb{R}^d/\mathbb{Z}^d$, $d\in\mathbb{N}^*$, $U_k(0)=0$) decays exponentially fast with the number of scales when the scale-ratios R_{k+1}/R_k are bounded above and below. From this we deduce the anomalous slow behavior for a Brownian Motion in a potential obtained as a superposition of an infinite number of scales: $dy_t=d\omega_t-\nabla V^\infty(y_t)dt$

Contents

AMS 1991 *Subject Classification.* Primary 74Q20, 60J60; secondary 35B27, 74Q10, 60F05, 31C05, 35B05.

Key words and phrases. Multi scale homogenization, anomalous diffusion, diffusion in fractal media, quantitative homogenization, quantitative bound on effective properties, renormalization with bounded scale ratios

*gerard.benarous@epfl.ch, DMA, EPFL, CH-1015 Lausanne , Switzerland

†houman.owhadi@epfl.ch, DMA, EPFL, CH-1015 Lausanne , Switzerland

1 Introduction

Homogenization in the presence of a large number of spatial scales is both very important for applications and far from understood from a mathematical standpoint. In the asymptotic regime where the spatial scales separate, i.e. when the ratio between successive scales tends to infinity, multi-scale homogenization is now well understood. See for instance Bensoussan-Lions-Papanicolaou [12], or Avellaneda [4], Allaire - Briane [2], [31], Jikov - Kozlov [28] or Avellaneda - Majda [6].

Nevertheless the case of multi-scale homogenization when spatial scales are not clearly separated, i.e. when the ratios between scales stay bounded, has been recognized as difficult and important. For instance, Avellaneda [5] (page 267) emphasizes that "the assumption of scale separation invoked in homogenization is not adequate for treating the most general problems of transport and diffusion in self-similar random media".

The potential use of multi-scale homogenization estimates for applications are numerous (see for instance [42] for applications to geology, or [15], [34], [17] for applications to Differential Effective Medium Theories). The main application of this line of ideas is perhaps to proving super-diffusivity for turbulent diffusion: see for instance [5], [7]; [22], [23], [24], [25], [45], [27], [20],[21], [18]; [13], [14]; [19] or [30].

We are here interested in sub-diffusivity problems. Consider the Brownian motion in a periodic potential, i.e. the diffusion process

$$dy_t = d\omega_t - \nabla V(y_t)dt \tag{1.1}$$

where V is periodic and smooth. It is a basic and simple fact of homogenization theory that y_t behaves in large times like a Brownian motion slower than the Brownian Motion ω_t driving the equation, i.e. $y^\epsilon(t) = \epsilon y_{t/\epsilon^2}$ converges in law to a Brownian motion with diffusivity matrix $D(V) < I_d$.

We first treat here the case where V is a periodic n-scale potential with ratios (between successive scales) bounded uniformly on n.We introduce a new approach which enables us to show exponential decay of the effective diffusivity matrix when the number of spatial scales grows to infinity.

From this exponential decay we will deduce the anomalous slow behavior of Brownian motions in potential V, when V is a superposition of an infinite number of scales.

We have studied this question with a particular application in mind, i.e. to prove that one of the basic mechanisms of anomalous slow diffusion in complex media is the existence of a large number of spatial scales, without a clear separation between them. This phenomenon has been attested for very regular self-similar fractals (see Barlow and Bass [10] and Osada [37] for the Sierpinski carpet) (see also [26]). Our goal is to implement rigorously the idea that the key for the sub-diffusivity is a never-ending or perpetual homogenization phenomenon over an infinite number of scales, the point being that our model will not have any self similarity or local symmetry hypotheses.

Our approach gives naturally much more detailed information in dimension one and this is the subject of [39].

This approach will be shown in forthcoming works to also give a proof of super-diffusive behavior for diffusion in some multi-scale divergence free fields (see [11] for the simple

case of shear flow and [40] for a general situation).
The second section contains the description of our model; the third one, the statement of our results and the fourth one the proofs.

2 The multi-scale medium

For $U \in L^\infty(\mathbb{T}_R^d)$ (we note $\mathbb{T}_R^d := R\mathbb{T}^d$), let m_U be the probability measure on $\mathbb{T}^d$ defined by

$$m_U(dx) = e^{-2U(x)}dx / \int_{\mathbb{T}_R^d} e^{-2U(x)}dx \tag{2.1}$$

The effective diffusivity $D(U)$ is the symmetric positive definite matrix given by

$${}^t l D(U) l = \inf_{f \in C^\infty(\mathbb{T}_R^d)} \int_{\mathbb{T}_R^d} |l - \nabla f(x)|^2 m_U(dx) \tag{2.2}$$

for l in $\mathbb{S}^{d-1}$ (the unit sphere of $\mathbb{R}^d$). Our purpose in this work is to obtain quantitative estimates for the effective diffusivity matrix of multi-scale potentials V_0^n given by a sum of periodic functions with (geometrically) increasing periods:

$$V_0^n = \sum_{k=0}^n U_k(\frac{x}{R_k}) \tag{2.3}$$

In this formula we have two important ingredients: the potentials U_k and the scale parameters R_k. We will now describe the hypothesis we make on these two items of our model.

1. Hypotheses on the potentials U_k
   We will assume that

$$U_k \in C^\alpha(\mathbb{T}^d) \tag{2.4}$$

$$U_k(0) = 0 \tag{2.5}$$

   Here $C^\alpha(\mathbb{T}^d)$ denotes the space of α-Holder continuous on the torus $\mathbb{T}^d$, with $0 < \alpha \leq 1$. We will also assume that the C^α-norm of the U_k are uniformly bounded, i.e.

$$K_\alpha := \sup_{k \in \mathbb{N}} \sup_{x \neq y} |U_k(x) - U_k(y)| / |x - y|^\alpha < \infty \tag{2.6}$$

   We will also need the notation

$$K_0 := \sup_{k \in \mathbb{N}} \operatorname{Osc}(U_k) \tag{2.7}$$

   where the oscillation of U_k is given by $\operatorname{Osc}(U) := \sup U - \inf U$.
   We also assume that the effective diffusivity matrices of the U_k's are uniformly bounded. Let $\lambda_{\min}(D(U_k))$ (respectively $\lambda_{\max}(D(U_k))$) be the smallest and largest eigenvalues of the effective diffusivity matrix $D(U_k)$. We will assume that

$$\lambda_{\min} := \inf_{k \in \mathbb{N}, l \in S^d} {}^t l D(U_k) l > 0 \tag{2.8}$$

$$\lambda_{\max} := \sup_{k \in \mathbb{N}, l \in S^d} {}^t l D(U_k) l < 1 \tag{2.9}$$

2. Hypotheses on the scale parameters R_k
   R_k is a spatial scale parameter growing exponentially fast with k, more precisely we will assume that $R_0 = r_0 = 1$ and that the ratios between scales defined by

$$r_k = R_k / R_{k-1} \in \mathbb{N}^* \tag{2.10}$$

   for $k \geq 1$, are integers uniformly bounded away from 1 and ∞: we will denote by

$$\rho_{\min} := \inf_{k \in \mathbb{N}^*} r_k \quad \text{and} \quad \rho_{\max} := \sup_{k \in \mathbb{N}^*} r_k \tag{2.11}$$

   and assume that

$$\rho_{\min} \geq 2 \quad \text{and} \quad \rho_{\max} < \infty \tag{2.12}$$

As an example, we have illustrated in the figure 1 the contour lines of $V_0^2(x,y) = \sum_{k=0}^2 U(\frac{x}{\rho^k}, \frac{y}{\rho^k})$, with $\rho = 4$ and $U(x,y) = \cos(x + \pi \sin(y) + 1)^2 \sin(\pi \cos(x) - 2y + 2) \cos(\pi \sin(x) + y)$

3 Main results

3.1 Quantitative estimates of the multi-scale effective diffusivity

3.1.1 The central estimate

Our first objective is to control the minimal and maximal eigenvalues of $D(V_0^n)$. More precisely writing I_d the $d \times d$ identity matrix we will prove that

Theorem 3.1. *Under the hypotheses (2.6), (2.10) and $\rho_{\min}^\alpha \geq K_\alpha$ there exists a constant C depending only on d, α, K_α, K_0 such that for all $n \geq 1$*

$$I_d \, e^{-n\epsilon} \prod_{k=0}^n \lambda_{\min}\big(D(U_k)\big) \leq D(V_0^n) \leq I_d \, e^{n\epsilon} \prod_{k=0}^n \lambda_{\max}(D(U_k)) \tag{3.1}$$

where

$$\epsilon = C \rho_{\min}^{-\alpha/2} \tag{3.2}$$

In particular ϵ tends to 0, when $\rho_{\min} \to \infty$

Remark 3.1. One can interpret this theorem as follows: $D(V_0^n)$ is bounded from below and from above by the bounds given by reiterated homogenization under the assumption of complete separation between scales, i.e. $\rho_{\min} \to \infty$ (product of minimal and maximal eigenvalues) times an error term $e^{n\epsilon}$ created by the *interaction* or *overlap* between the different scales.

Remark 3.2. Originally the problem of estimating $D(V_0^n)$ was called for in connection with applied sciences, and heuristic theories such as Differential Effective Medium theory have been developed for that purpose. This theory (DEM theory) models a two phase composite by incrementally adding inclusions of one phase to a background matrix of the other and then recomputing the new effective background material at each increment [15], [34], [17]. It was first proposed by Bruggeman to compute the conductivity of a two-component composite structure formed by successive substitutions ([16] and [1]) and

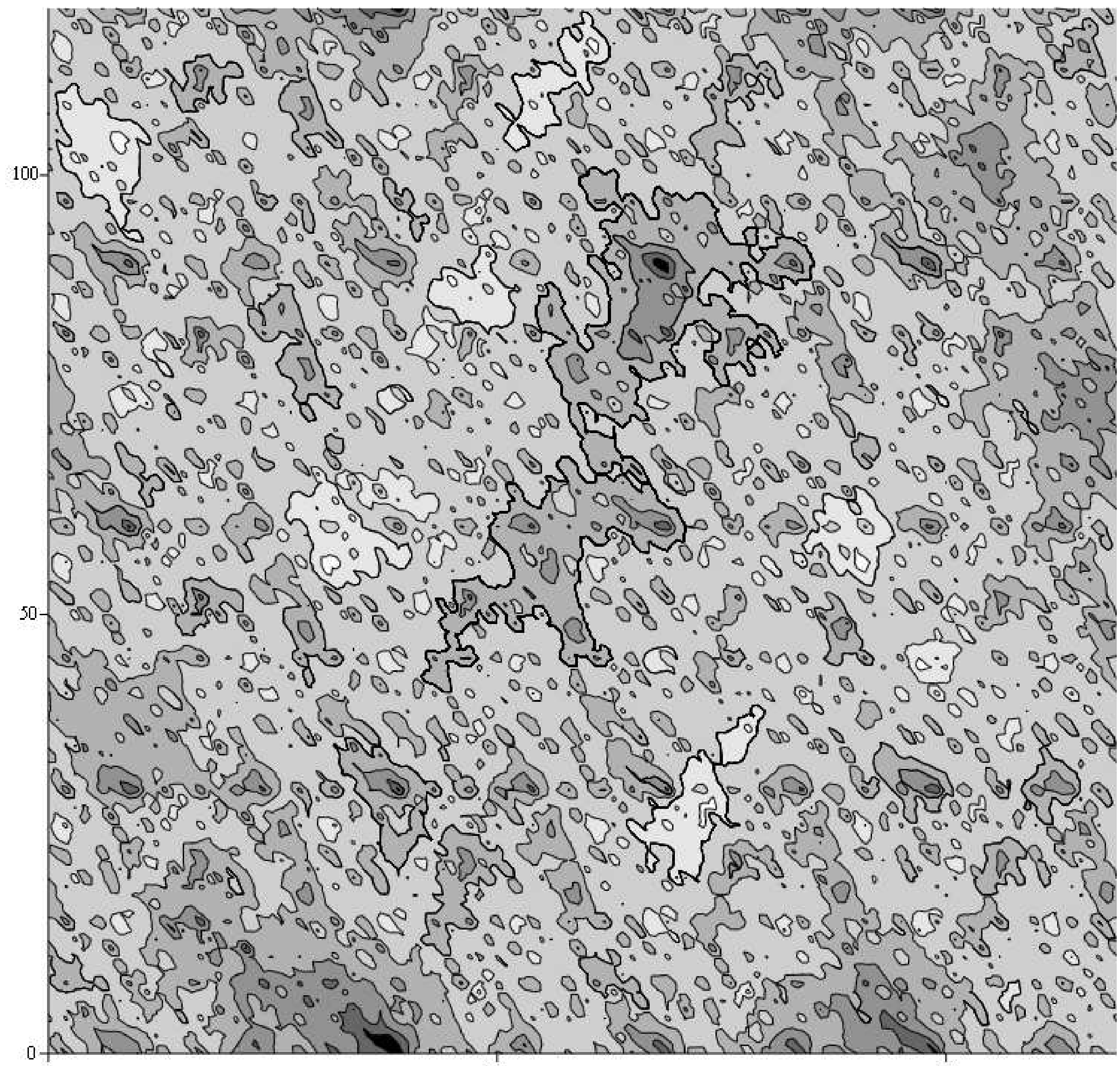

Figure 1: A particular case

generalized by Norris [35] to materials with more than two phases.
More recently Avellaneda [4] has given a rigorous interpretation of the equations obtained by DEM theories showing that they are *homogeneous limit equations* with two very important features: complete separation of scales and "dilution of phases". That is to say, each "phase" U_k is present at an infinite number of scales in a homogeneous way. Yet two different phases never interact because they always appear at scales whose ratio is ∞. Moreover the macroscopic influence of each phase is totally (but non uniformly) diluted in the infinite number of scales at which it appears. In our context, complete separation of scales would mean that R_{k+1}/R_k grows sufficiently fast to ∞, and "dilution of phases" would bmean that $V_0^n = \sum_{k=0}^{n} U_k^n(x/R_k)$ with $U_k^n \to 0$ as $n \to \infty$. The rigorous tool used by Avellaneda to obtain this interpretation is reiterated homogenization [12].
May be the most recent work on this topic is the article by Jikov-Kozlov [28], who work under the assumption of "dilution of phases" and fast separation between scales, more precisely under the condition that $\sum_{k=1}^{\infty} k\big(R_k/R_{k+1}\big)^2 < \infty$. Jikov-Kozlov use the classical toolbox of asymptotic expansion, plugging well chosen test functions in the cell problem. This method of asymptotic expansion is simply not at all available in our context.

Theorem 3.1 will be proved by induction on the number of scales. The basic step in this induction is the following estimate (3.4) on the effective diffusivity for a two-scale periodic medium.
Let $U, T \in C^\alpha(\mathbb{T}^d)$. Let us define for $R \in \mathbb{N}^*$, $S_R U \in C^\alpha(\mathbb{T}^d)$ by $S_R U(x) = U(Rx)$. We will need to estimate $D(S_R U + T)$ the effective diffusivity for a two-scale medium when R is a large integer. Let us define $D(U,T)$, the symmetric definite positive matrix given by

$$ {}^t l D(U,T) l = \inf_{f \in C^\infty(\mathbb{T}_1^d)} \int_{\mathbb{T}_1^d} {}^t(l - \nabla f(x)) D(U) (l - \nabla f(x)) m_T(dx) \quad \text{for } l \in \mathbb{R}^d \tag{3.3} $$

Theorem 3.2. *Let $R \in \mathbb{N}^*$ and $U, T \in C^\alpha(\mathbb{T}^d)$. If $R^\alpha \geq \|T\|_\alpha$ then there exists a constant C depending only on $d, \mathrm{Osc}(U), \|U\|_\alpha, \alpha$ such that*

$$ e^{-\epsilon} D(U,T) \leq D(S_R U + T) \leq D(U,T) e^{\epsilon} \tag{3.4} $$

with $\epsilon = C R^{-\alpha/2}$

Remark 3.3. Theorem 3.2 implies obviously that

$$ D(U,T) = \lim_{R \to \infty} D(S_R U + T) \tag{3.5} $$

so that $D(U,T)$ should be interpreted as the effective diffusivity of the two scale medium for a complete separation of scales. Naturally $D(U,T)$ is also computable from an explicit cell problem (see (4.12)).

Remark 3.4. The estimate given in theorem 3.2 is stronger than needed for theorem 3.1. It gives a control of $D(S_R U + T)$ in terms of $D(U,T)$ and not only of the minimal and maximal eigenvalues of $D(U)$ and $D(T)$. In fact we will only use its corollary 3.1 given below, which is deduced using the variational formulation (3.3).

Corollary 3.1. *Let $R \in \mathbb{N}^*$ and $U, T \in C^\alpha(\mathbb{T}^d)$. If $R^\alpha \geq \|T\|_\alpha$ then there exists a constant C depending only on $d, \mathrm{Osc}(U), \|U\|_\alpha, \alpha$ such that*

$$ \lambda_{\min}\big(D(U)\big) D(T) e^{-\epsilon} \leq D(S_R U + T) \leq \lambda_{\max}\big(D(U)\big) D(T) e^{\epsilon} \tag{3.6} $$

with $\epsilon = C R^{-\alpha/2}$

Remark 3.5. We mentioned that theorem 3.1 is proved by induction. This induction differs from the one used in reiterated homogenization or DEM theories by the fact we homogenize on the larger scales first and add at each step a smaller scale.

Let us introduce the following upper and lower exponential rates

Definition 3.1.

$$\lambda^+ = \lim\sup_{n\to\infty} \frac{1}{n} \ln \lambda_{\max}\big(D(V_0^n)\big) \tag{3.7}$$

$$\lambda^- = \lim\inf_{n\to\infty} \frac{1}{n} \ln \lambda_{\min}\big(D(V_0^n)\big) \tag{3.8}$$

Theorem 3.1 implies the exponential decay of $D(V_0^{n-1})$, i.e.

Corollary 3.2. *Under the hypotheses (2.6), (2.10) and $\rho_{\min}^{\alpha} \geq K_\alpha$ one has (with ϵ given by (3.2)) for $n \geq 1$*

$$I_d e^{-n\epsilon}\lambda_{\min}^{n+1} \leq D(V_0^n) \leq I_d e^{n\epsilon}\lambda_{\max}^{n+1} \tag{3.9}$$

and

$$\lambda^+ \leq \ln \lambda_{\max} + \epsilon \tag{3.10}$$

$$\lambda^- \geq \ln \lambda_{\min} - \epsilon \tag{3.11}$$

In particular if $\lambda_{\max} < 1$ then there exists a constant $\rho_0 = \big(1 + C_{d,K_0,K_\alpha,\alpha}/(-\ln \lambda_{\max})\big)^{\frac{2}{\alpha}}$ such that, for $\rho_{\min} \geq \rho_0$

$$\lambda^+ < 0 \tag{3.12}$$

Thus one obtains the exponential decay of $D(V^n)$ only for a minimal separation between scales, i.e. ρ_{min} greater than a constant ρ_0 characterized by the medium. It is natural to wonder whether this condition is necessary and what happens below this constant ρ_0. We will give a partial answer to that question, in the simple case when the medium V is self-similar. We will see that it is possible to find models such that for a certain value C of the separation parameter $\rho_{\min} = C$, $D(V_0^n)$ decays exponentially and for $\rho_{\min} = C + 1$, $D(V_0^n)$ stays bounded away from zero. This will be done using a link with large deviation theory.

3.1.2 The self similar case

Definition 3.2. The medium V is called self similar if and only if $\forall n$, $U_n = U$ and $R_n = \rho^n$ with $\rho \in \mathbb{N}$, $\rho \geq 2$

Definition 3.3. For $U \in C^\alpha(\mathbb{T}^d)$ and $\rho \in \mathbb{N}/\{0,1\}$ we denote by $p_\rho(U)$ the pressure associated to the shift $s_\rho(x) = \rho x$ on $\mathbb{T}^d$, i.e.

$$p_\rho(U) = \sup_\mu \Big(\int_{\mathbb{T}^d} U(x)d\mu(x) + h_\rho(\mu)\Big) \tag{3.13}$$

where h_ρ is the Komogorov-Sinai entropy related to the shift s_ρ. We denote

$$P_\rho(U) = p_\rho(U) - p_\rho(0) = p_\rho(U) - d\ln R \tag{3.14}$$

We refer to [29] and [41] for a reminder on the pressure, let us observe that $\mathcal{P}_\rho(0)$ differs from the standard definition of the topological pressure by the constant $d\ln\rho$ so that $\mathcal{P}_\rho(0) = 0$.
We will relate in the self-similar case the exponential rates λ^+ and λ^- to pressures for the shift s_ρ and to large deviation at level 3 for i.i.d. random variables.
In the self similar case we will write $\lambda^-(-U)$ the exponential rates associated to $D(-V_0^n)$.
We write

$$Z(U) = -\Big(\mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U)\Big) \tag{3.15}$$

Theorem 3.3. *If the medium V is self similar then*

1. *If $d = 1$*

$$\lambda^+(U) = \lambda^-(U) = Z(U) \tag{3.16}$$

2. *if $d = 2$ then*

$$\lambda^+(U) + \lambda^-(-U) = Z(U) \tag{3.17}$$

Moreover if there exists an isometry A of $\mathbb{R}^d$ such that $U(Ax) = -U(x)$ and a reflection B such that $U(Bx) = U(x)$ then $\lambda^-(-U) = \lambda^-(U) = \lambda^+(U)$ so that

$$\lambda^+(U) = \lambda^-(U) = Z(U)/2 \tag{3.18}$$

3. *For any d*

$$Z(U) \le \lambda^-(U) \le \lambda^+(U) \le 0 \tag{3.19}$$

Remark 3.6. The statement (3.16) is obtained from the explicit formula for $D(V_0^n)$ in $d = 1$, see [39].

Remark 3.7. It is obviously important to know when $Z(U)$ is strictly negative to be able to use this theorem. A well known and useful criterion can be stated as $Z(U) < 0$ if and only if U does not belong to the closure of the vector space spanned by cocycles, which can be shown to be equivalent to say that

$$Z(U) < 0 \Leftrightarrow \limsup_{n\to\infty} \frac{1}{n}\Big\|\sum_{k=0}^{n-1}\Big(U(\rho^k x) - \int_{\mathbb{T}^d} U(x)dx\Big)\Big\|_\infty > 0 \tag{3.20}$$

We refer to [39] for the proof of the last statement.

Example 3.1. Let $U(x) = \sin(x) - \sin(81x)$ in dimension one. In fact (3.16) and (3.20) shows that $\lambda^+(U) < 0$ as soon as $\rho \ge 82$. For $\rho \le 81$ the situation is a bit surprising: $\lambda^+(U) < 0$ for $\rho \ne 3, 9, 81$. For these exceptional values $\lambda^+(U) = 0$ and in fact $D(V_0^n)$ remains lower bounded by a strictly positive constant.
This example shows that for a given potential U, even though the multi-scale effective diffusivity $D(V_0^n)$ decays exponentially for ρ large enough, one can find isolated values of the scale parameter for which $D(V_0^n)$ remains bounded from below.

Remark 3.8. The symmetry hypotheses given in theorem 3.3.2 are only used to prove that for all n, $D(V_0^n) = D(-V_0^n)$ and $\lambda_{\max}\big(D(V_0^n)\big) = \lambda_{\min}\big(D(V_0^n)\big)$ (see proposition 4.1)

3.2 Sub-diffusive behavior from homogenization on infinitely many scales

Here we consider the diffusion process given by the Brownian Motion in the potential

$$V = V_0^\infty = \sum_{k=0}^{\infty} U_k(x/R_k) \tag{3.21}$$

We assume in this section that the hypotheses (2.4) , (2.5), (2.6), (2.8), (2.9), (2.10) and (2.12) hold. To start with we will assume that

$$\alpha = 1 \quad \text{and that the potentials } U_k \text{ are uniformly } C^1. \tag{3.22}$$

In particular V is well defined and belongs to $C^1(\mathbb{R}^d)$ and $\|\nabla V\|_\infty < \infty$.
The diffusion process associated to the potential V is well defined by the Stochastic Differential Equation

$$dy_t = d\omega_t - \nabla V(y_t)dt \tag{3.23}$$

We will show that the multi-scale structure of V can lead to an anomalous slow behavior for the process y_t. To describe this sub-diffusive phenomenon we choose to compute the mean exit time from large balls, i.e. Let

$$\tau(r) = \inf\{t > 0 \,:\, |y_t| \geq r\} \tag{3.24}$$

We would like to show that $\mathbb{E}_x[\tau(r)]$ grows faster than quadratic in r when $r \to \infty$ uniformly in x. We cannot obtain such pointwise results in dimension $d > 1$ (see 3.2.1 for a discussion, the case $d = 1$ is treated in [39]). But we will start with averaged results on those mean exit times.
The fact that the homogenization results of subsection 3.1 can be of some help to estimate the mean exit times is shown by the following lemma

Lemma 3.1. *For $U \in C^\infty(\mathbb{T}^d_R)$, $(R > 0)$ writing $\mathbb{E}^U$ the exit times associated to the diffusion generated by $L_U = \Delta/2 - \nabla U \nabla$ one has*

$$\begin{aligned} \mathbb{E}^U_x[\tau(x,r)] &\leq C_2 \frac{r^2}{\lambda_{\max}\big(D(U)\big)} + C_d e^{(9d+15)\,\mathrm{Osc}(U)} R^2 \\ &\geq C_1 \frac{r^2}{\lambda_{\max}\big(D(U)\big)} - C_d e^{(9d+15)\,\mathrm{Osc}(U)} R^2 \end{aligned} \tag{3.25}$$

Let $m_{V,r}$ be the probability measure on the ball $B(0,r)$ given by

$$m_{V,r}(dx) = e^{-2V(x)}\, dx \Big/ \Big(\int_{B(0,r)} e^{-2V(x)}\, dx \Big) \tag{3.26}$$

We will consider the mean exit time for the process started with initial distribution $m_{V,r}$, i.e.

$$\mathbb{E}_{m_{V,r}}\big[\tau(r)\big] = \int_{B(0,r)} \mathbb{E}_x\big[\tau(r)\big]\, m_{V,r}(dx) \tag{3.27}$$

Theorem 3.4. *Under the hypothesis $\lambda_{\max} < 1$ there exists C_2 depending on $d, \lambda_{\max}, K_0, K_\alpha, \alpha$ such that if $\rho_{\min} > C_2$ then*

$$\liminf_{r\to\infty} \frac{\ln \mathbb{E}_{m_{V,r}}\big[\tau(r)\big]}{\ln r} > 2 \tag{3.28}$$

More precisely there exists $C_3 > 0$, $C_4 > 0$, $C_5 > 0$ such that for $r > C_3$,

$$\mathbb{E}_{m_{V,r}}\big[\tau(r)\big] = r^{2+\nu(r)} \tag{3.29}$$

with

$$0 < C_4 < \frac{\ln \frac{1}{\lambda_{\max}}}{\ln \rho_{\max}}\big(1 - \frac{C_5}{\ln \rho_{\min}}\big) - \frac{1}{\ln r} C_5 \leq \nu(r) \leq \frac{\ln \frac{1}{\lambda_{\min}}}{\ln \rho_{\min}}\big(1 + \frac{C_5}{\ln \rho_{\min}}\big) + \frac{1}{\ln r} C_5 \tag{3.30}$$

Where C_3 and C_5 depend on $(d, K_0, K_\alpha, \alpha)$ and C_4 on $(\lambda_{\max}, \rho_{\max})$.

The proof of this result relies heavily on theorem 3.1. The idea being that $\mathbb{E}_{m_{V,r}}\big[\tau(r)\big]$ is close, when r is large to $r^2/\lambda_{\max}\big(D(V_0^n)\big)$ where n is roughly $\sup\{m \in \mathbb{N} : R_m \leq r\}$. So that the exponential decay of $D(V_0^n)$ gives the super-quadratic behavior of $\mathbb{E}_{m_{V,r}}\big[\tau(r)\big]$, i.e. sub-diffusivity.

Remark 3.9. The differentiability hypothesis (3.22) though convenient in order to define the process y_t as a solution of the SDE 3.23 is in fact useless. The theorem is also meaningful and true with $0 < \alpha < 1$. See section 4.2 for an explanation.

3.2.1 Pointwise estimates on the anomaly

Theorem 3.4 gives the anomalous behavior of the exit times with respect to the invariant measure of the diffusion and it is desirable to seek for pointwise estimates of this anomaly. The additional difficulty is to obtain quantitative estimates on the stability of divergence form elliptic operators under a perturbation of their principal parts (see conjecture 3.1). By stability we mean here the validity of the following condition 3.1.
For $U \in C^1(\overline{B(z,r)})$. Write $\mathbb{E}^U$, the expectation associated to the diffusions generated by $L_U = \frac{1}{2}\Delta - \nabla U \nabla$. V is said to satisfy the stability condition 3.1 if and only if:

Condition 3.1. *There exists $\mu > 0$ such that for all $n \in N$, all $z \in \mathbb{R}^d$, and all $r > 0$,*

$$\frac{1}{\mu} e^{-\mu \operatorname{Osc}_{B(z,r)}(V_{n+1}^\infty)} \inf_{x \in B(z,\frac{r}{2})} E_x^{V_0^n}\big[\tau(B(z,r))\big] \leq E_z^V\big[\tau(B(z,r))\big] \tag{3.31}$$

$$E_z^V\big[\tau(B(z,r))\big] \leq \mu e^{\mu \operatorname{Osc}_{B(z,r)}(V_{n+1}^\infty)} \sup_{x \in B(z,r)} E_x^{V_0^n}\big[\tau(B(z,r))\big] \tag{3.32}$$

Where $\operatorname{Osc}_{B(z,r)}(U)$ stands for $\sup_{B(z,r)} U - \inf_{B(z,r)} U$. Under the stability condition 3.1, we can obtain sharp pointwize estimates on the mean exit times.

Theorem 3.5. *If V satisfies the stability condition 3.1, then there exist a constant C_6 depending on $(d, K_0, K_\alpha, \alpha, \mu, \lambda_{\max})$ such that for $\rho_{\min} > C_6$, one has for all $x \in \mathbb{R}^d$*

$$\liminf_{r\to\infty} \frac{\ln \mathbb{E}_x\big[\tau(B(x,r))\big]}{\ln r} > 2 \tag{3.33}$$

More precisely there exists a function $\sigma(r)$ such that for $r > C_7$ one has

$$C_8 r^{2+\sigma(r)(1-\gamma)} \leq \mathbb{E}_x\big[\tau(B(x,r))\big] \leq C_9 r^{2+\sigma(r)(1+\gamma)} \tag{3.34}$$

with

$$\frac{\ln\frac{1}{\lambda_{\max}}}{\ln\rho_{\max}}(1+\frac{C_3}{\ln\rho_{\min}})^{-1}\leq\sigma(r)\leq\frac{\ln\frac{1}{\lambda_{\min}}}{\ln\rho_{\min}}(1+\frac{C_4}{\ln\rho_{\min}}) \tag{3.35}$$

and $\gamma = C_5K_0/(\ln\rho_{\min}) < 0.5$. *Where the constants* C_3, C_4, C_7, C_8 *and* C_9 *depend on* $(d, K_0, K_\alpha, \alpha, \mu, \lambda_{\max})$ *and* C_5 *on* d.

Here $\tau(B(x,r))$ denotes the exit time from the ball $B(x,r)$.

Remark 3.10. In fact $\sigma(r)$ can be described rather precisely. Let,

$$\sigma(r,n) = -\ln\lambda_{\max}D(V_0^n)\Big/\ln r \tag{3.36}$$

Define $n_{ef}(r, C_1, C_2) = \sup\{n \geq 0 \ : \ e^{(n+1)C_1K_0}R_n^2 \leq C_2r^2\}$. Then there exists C_1, C_2 depending only on d such that $\sigma(r)$ in theorem 3.5 is $\sigma\big(r, n_{ef}(r, C_1, C_2)\big)$.

Using the precise information of theorem 3.5 we can estimate the tails of probability transitions for the process y_t (or the tail of the heat kernel for the operator L_V). We get non-Gaussian upper bound similar to the (more precise) ones proved for fractal diffusions (see [10] and [26])

Theorem 3.6. *If* V *satisfies the stability condition 3.1, then for* $\rho_{\min} > C_6$*,* $r > 0$

$$C_{10}r \leq t \leq C_{11}r^{2+\sigma(r)(1-3\gamma)} \tag{3.37}$$

one has

$$\ln\mathbb{P}_x[|y_t - x| \geq r] \leq -C_{13}\frac{r^2}{t}(\frac{t}{r})^{\nu(t/h)} \tag{3.38}$$

with ($C_{17} < 0.5\ln\rho_{\min}$)

$$0<\frac{\ln\frac{1}{\lambda_{\max}}}{\ln\rho_{\max}}(1-\frac{C_{14}}{\ln\rho_{\min}})\leq\nu(y)\leq\frac{\ln\frac{1}{\lambda_{\min}}}{\ln\rho_{\min}}(1-\frac{C_{15}}{\ln\rho_{\min}}) \tag{3.39}$$

Where C_6, C_{13}, C_{14} *and* C_{15} *depend on* $(d, K_0, K_\alpha, \alpha, \mu, \lambda_{\max})$*;* C_{17} *on* d, K_0*;* C_{10}, C_{11} *on* $(d, K_0, K_\alpha, \alpha, \mu, \lambda_{\max}, \rho_{\min}, \rho_{\max})$

Remark 3.11. The non Gaussian structure of (3.38) is similar to the one obtained for diffusion processes in fractals. Indeed

$$-C\frac{h^2}{t}(\frac{t}{h})^{\nu} = -C\big(\frac{|x-y|^{d_w}}{t}\big)^{\frac{1}{d_w-1}} \tag{3.40}$$

with $d_w \sim 2+\nu$.

Next, it has been shown in [39] that for $U \in L^\infty(\mathbb{T}_R^d)$, $\ln p^U(x,y,t)$ is roughly $-{}^t(y-x)D^{-1}(U)(y-x)/t$ for $t > R|x-y|$ (homogenized behavior). Where p^U is the heat kernel associated to L_U. Next, writing $n_{ef}(t/h) = \sup_n\{R_n \leq t/h\}$ the number of scales that one can consider as homogenized in the estimation of the heat kernel tail one obtains from a heuristic computation (which can be made rigorous in dimension one, see [39]) that for $C_{10}h \leq t \leq C_{11}h^{2+\mu}$,

$$\ln\mathbb{P}(y_t \geq h) \leq -C\frac{h^2}{t\lambda^{n_{ef}(t/h)}} \sim -C\frac{h^2}{t}(\frac{t}{h})^{-\frac{\ln\lambda}{\ln\rho}} \sim -C\big(\frac{|x-y|^{d_w}}{t}\big)^{\frac{1}{d_w-1}} \tag{3.41}$$

with $d_w \sim 2 - \frac{\ln \lambda}{\ln \rho}$. The equation 3.41 suggests that the origin of the anomalous shape of the heat kernel for the reflected Brownian Motion on the Sierpinski carpet can be explained by a *perpetual homogenization* phenomenon and the formula linking the number of effective scales and the ratio t/h.
The condition $C_{10}h \leq t$ can be translated into "homogenization has started on at least the first scale" ($n_{ef} \geq 1$) and the second one $t \leq C_{11} h^{2+\mu}$ into "the heat kernel associated to L_V is far from its diagonal regime" (one can have $h^2/t << 1$ before reaching that regime, this is explained by the slow down of the diffusion).

The weak point of theorems 3.5 and 3.6 is naturally that checking condition 3.1 seems difficult. But we believe that in fact this condition is always true (we refer to the chapter 13 of [38]), since this condition is a consequence of the following conjecture (see [39], proposition 2.3).

Conjecture 3.1. *There exists a constant C_d depending only on the dimension of the space such that for $\lambda \in C^\infty(\overline{B(0,1)})$ such that $\lambda > 0$ on $\overline{B(0,1)}$ and $\phi, \psi \in C^2(\overline{B(0,1)})$ null on $\partial B(0,1)$ and both sub harmonic with respect to the operator $-\nabla(\lambda\nabla)$, one has*

$$\int_{B(0,1)} \lambda(x) |\nabla \phi(x).\nabla \psi(x)| \, dx \leq C_d \int_{B(0,1)} \lambda(x) \nabla \phi(x).\nabla \psi(x) \, dx \tag{3.42}$$

It is simple to see [39] that this conjecture is true in dimension one with $C_d = 3$. So proving conjecture 3.1 would give the pointwise estimates of theorem 3.5 and the tail estimate for the heat kernel in theorem 3.6.

Remark 3.12. Here we have assumed that the U_k are uniformly C_1 but let us observe that since theorems 3.5 and 3.6 are robust in their dependence on α and K_α (one can choose $\alpha < 1$). One can built a process, with the assumption that the U_k are α-Holder continuous, whose mean exit times and heat kernel tail verify the estimates given in theorems 3.5 and 3.6.

4 Proofs

4.1 Multi-scale homogenization with bounded ratios

4.1.1 Global estimates of the multi-scale effective diffusivity: theorem 3.1

The proof of theorem 3.1 will follow from the Corollary 3.1 by a simple induction. Let $n \in N/\{0,1\}$, $p \in \mathbb{N}, 1 \leq p \leq n$ and assume that

$$I_d e^{-(n-p)\epsilon(\rho_{\min})} \prod_{k=p}^{n} \lambda_{\min}\big(D(U_k)\big) \leq D(V_p^n) \leq I_d e^{(n-p)\epsilon(\rho_{\min})} \prod_{k=p}^{n} \lambda_{\max}(D(U_k)) \tag{4.1}$$

One pass from the quantitative control on $D(V_p^n)$ to a control on $D(V_{p-1}^n)$ by choosing $U(x) = U_{p-1}(x)$, $T(x) = V_p^n(R_n x)$ and $R = R_n/R_{p-1}$ in theorem 3.2 and observing that $\|T\|_\alpha / R^\alpha \leq (2^\alpha - 1)^{-1} K_\alpha / \rho_{\min}^\alpha$. This proves the induction and henceforth the theorem.

4.1.2 Quantitative multi-scale-homogenization: Upper bound in the theorem 3.2

4.1.2.1 We will use the notation introduced in theorem 3.2. By the variational formula (2.2), $D(U)$ is continuous with respect to U in L^∞-norm, it is sufficient to prove

theorem 3.2 assuming that U and T are smooth.
First let us prove that when homogenization takes place on two scales separated by a ratio R, the influence of a translation of the first one with respect to the second one on the global effective diffusivity can easily be controlled, i.e. for $y \in T_1^d$, writing Θ_y the translation operator $T(x) \to \Theta_y T(x) = T(x+y)$

Lemma 4.1.

$$e^{-4\frac{\|T\|_\alpha}{R^\alpha}} D(S_R U + T) \leq D(S_R \Theta_y U + T) \leq e^{4\frac{\|T\|_\alpha}{R^\alpha}} D(S_R U + T) \tag{4.2}$$

Proof. The proof follows by observing that $S_R U + \Theta_y T = \Theta_{[Ry]/R}(S_R U + T) + \Theta_y T - \Theta_{[Ry]/R} T$ where $[Ry]$ stands for the vector with the integral parts of $(yR)_i$ as coordinates. Thus by the variational definition of the effective diffusivity

$$D(S_R U + \Theta_y T) \leq e^{4\|\Theta_y T - \Theta_{[Ry]/R} T\|_\infty} D(\Theta_{[Ry]/R}(S_R U + T)) \tag{4.3}$$

and the equation (4.2) follows by observing that the effective diffusivity is invariant under a translation of the medium: $D(\Theta_{[Ry]/R}(S_R U + T)) = D(S_R U + T)$. □

Next we will obtain a quantitative control on $\int_{y\in\mathbb{T}^d} D(S_R U + \Theta_y T) dy$

Lemma 4.2. *For $R > \|T\|_\alpha$*

$$\int_{y\in\mathbb{T}^d} D(S_R U + \Theta_y T) dy \leq e^{22\frac{\|T\|_\alpha}{R^\alpha}} \big(1 + C_d e^{C_d \operatorname{Osc}(U)} (\|T\|_\alpha / R^\alpha)^{\frac{1}{2}}\big) D(U,T) \tag{4.4}$$

Let us observe that the combination of lemma 4.2 with 4.1 gives the upper bound (3.4) in theorem 3.2.
Write χ_l^U the solution of the cell problem associated to U. We remind that for $l \in \mathbb{R}^d$, $L_U = 1/2\Delta - \nabla U \nabla$, $L_U \chi_l = -l \nabla U$, $\chi_l^U(0) = 0$ and

$${}^t l D(U) l = \int_{\mathbb{T}^d} |l - \nabla \chi_l|^2 m_U(dx) = \int_{\mathbb{T}^d} {}^t(l - \nabla \chi_l).l m_U(dx) \tag{4.5}$$

Write $\chi^{D(U),T}$ the $\mathbb{T}^d$ periodic solution of the following cell problem (which corresponds to a complete homogenization on the smaller scale): for $l \in \mathbb{S}^{d-1}$

$$\nabla\big(e^{-2T} D(U)(l - \nabla \chi_l^{D(U),T})\big) = 0 \tag{4.6}$$

Write for $y \in \mathbb{T}^d$, $x \to \chi(x,y)$ the solution of the cell problem associated to $S_R \Theta_y U + T$. Let $l \in \mathbb{S}^{d-1}$, by the formula associating the effective diffusivity and the solution of the cell problem and using that $l - \nabla_x \chi_l(x,y)$ is harmonic with respect to $L_{S_R U + \Theta_y T}$, one obtains

$$\int_{y\in\mathbb{T}^d} {}^t l D(S_R \Theta_y U + T) l dy = \int_{\mathbb{T}^d \times \mathbb{T}^d} (l - \nabla_x \chi_l(x,y)).l \, dx \, dy \tag{4.7}$$

Writing the decomposition

$$l = (I_d - \nabla \chi^U_.(Rx+y))(l - \nabla \chi_l^{D(U),T}(x)) + \nabla \chi^U_.(Rx+y)(\nabla \chi_l^{D(U),T}(x) - l) m_{U(R.x+y)+T(.)}(dx) \, dy \tag{4.8}$$

we get that

$$\int_{y\in\mathbb{T}^d} {}^t l D(S_R \Theta_y U + T) l dy = I_1 - I_2 \tag{4.9}$$

With

$$I_1 = \int_{\mathbb{T}^d\times\mathbb{T}^d} (l - \nabla_x \chi_l(x,y))(I_d - \nabla\chi_.^U(Rx+y))(l - \nabla\chi_l^{D(U),T}(x)) m_{U(R.+y)+T(.)}(dx)\,dy \tag{4.10}$$

and

$$I_2 = \int_{\mathbb{T}^d\times\mathbb{T}^d} (l - \nabla_x \chi_l(x,y))\nabla\chi_.^U(Rx+y)(\nabla\chi_l^{D(U),T}(x) - l) m_{U(Rx+y)+T(.)}(dx)\,dy \tag{4.11}$$

It is easy to see that $\chi_.^{D(U),T}$ is a minimizer in the variational formula (3.3) associated to $D(U,T)$, which is the effective diffusivity corresponding to two-scale homogenization on U,T with complete separation between the scales, that is to say:

$$D(U,T) = \int_{x\in\mathbb{T}^d} {}^t(I_d - \nabla\chi_.^{D(U),T}(x))D(U)(I_d - \nabla\chi_.^{D(U),T}(x))m_T(dx) \tag{4.12}$$

A simple use of Cauchy-Schwarz inequality gives an upper bound on I_1,

$$\begin{aligned} I_1 \leq & \Big(\int_{(x,y)\in(\mathbb{T}^d)^2} |l - \nabla_x\chi_l(x,y)|^2 m^{U(Rx+y)+T(x)}(dx)\,dy\Big)^{\frac{1}{2}} \\ & \times\Big(\int_{(x,y)\in(\mathbb{T}^d)^2} |(I_d - \nabla\chi_.^U(Rx+y))(l - \nabla\chi_l^{D(U),T}(x))|^2 m^{U(Rx+y)+T(x)}(dx), dy\Big)^{\frac{1}{2}} \end{aligned} \tag{4.13}$$

Integrating first in y in the second term and, using the formulas linking effective diffusivities and solutions of the cell problem, we obtain

$$I_1 \leq \Big(\int_{y\in\mathbb{T}^d} {}^tlD(S_R\Theta_yU + T)l\,dy\Big)^{\frac{1}{2}} \times \Big({}^tlD(U,T)l\Big)^{\frac{1}{2}} e^{\frac{\|T\|_\alpha}{R^\alpha}} \tag{4.14}$$

We now estimate I_2. The following lemma together with (4.9) and (4.14) gives lemma 4.2.

Lemma 4.3.

$$\begin{aligned} |I_2| \leq & \Big(\int_{y\in\mathbb{T}^d} {}^tlD(S_R\Theta_yU + T)l\,dy\Big)^{\frac{1}{2}} \times \Big({}^tlD(U,T)l\Big)^{\frac{1}{2}} \\ & C_d e^{C_d \operatorname{Osc}(U)} e^{4\frac{\|T\|_\alpha}{R^\alpha}}(e^{8\frac{\|T\|_\alpha}{R^\alpha}} - 1)^{\frac{1}{2}} \end{aligned} \tag{4.15}$$

The proof of this lemma relies heavily on the following elliptic type estimate

Lemma 4.4.

$$\|\chi_l^U\|_\infty \leq C_d \exp\big((3d+2)\operatorname{Osc}(U)\big)|l| \tag{4.16}$$

This lemma is a consequence of theorem 5.4, chapter 5 of [43] on elliptic equations with discontinuous coefficients(see also [44]), we give the proof of lemma 4.4 for the sake of completeness in paragraph 4.1.2.2.

We will now prove lemma 4.3. First we will estimate the distance between $\chi_l(x,y)$ and

$\chi_l(x+y/R,0)$ for $y\in[0,1]^d$ in H^1 norm.
By the orthogonality property of the solution of the cell problem for $y\in[0,1]^d$

$$\begin{aligned}
&\int_{x\in\mathbb{T}^d}|\nabla_x\chi_l(x+\frac{y}{R},0)-\nabla_x\chi_l(x,y)|^2\frac{e^{-2(U(Rx+y)+T(x))}}{\int_{\mathbb{T}^d}e^{-2(U(Rz+y)+T(z))}dz}dx\,dy\\
&\quad=\int_{x\in\mathbb{T}^d}|l-\nabla_x\chi_l(x+\frac{y}{R},0)|^2\frac{e^{-2(U(Rx+y)+T(x))}}{\int_{\mathbb{T}^d}e^{-2(U(Rz+y)+T(z))}dz}dx\,dy-{}^tlD(S_R\Theta_yU+T)l\\
&\quad\leq{}^tlD(S_RU+T)le^{4\frac{\|T\|_\alpha}{R^\alpha}}-{}^tlD(S_R\Theta_yU+T)l
\end{aligned}\tag{4.17}$$

Thus by lemma 4.1, for $y\in[0,1]^d$,

$$\begin{aligned}
&\int_{x\in\mathbb{T}^d}|\nabla_x\chi_l(x+\frac{y}{R},0)-\nabla_x\chi_l(x,y)|^2m_{U(R.x+y)+T(.)}(dx)\,dy\\
&\qquad\qquad\leq{}^tlD(S_R\Theta_yU+T)l(e^{8\frac{\|T\|_\alpha}{R^\alpha}}-1)
\end{aligned}\tag{4.18}$$

Let us introduce

$$\begin{aligned}
I_3=&\int_{(x,y)\in\mathbb{T}^d\times[0,1]^d}(l-\nabla_x\chi_l(x+\frac{y}{R},0))\nabla\chi_.^U(Rx+y)(\nabla\chi_l^{D(U),T}(x)-l)\\
&\frac{e^{-2(U(Rx+y)+T(x+\frac{y}{R}))}}{\int_{\mathbb{T}^d}e^{-2U(z)}dz\int_{\mathbb{T}^d}e^{-2T(z)}dz}dx\,dy
\end{aligned}\tag{4.19}$$

one has

$$\begin{aligned}
|I_2-I_3|\leq&\Big(\int_{y\in\mathbb{T}^d}{}^tlD(S_R\Theta_yU+T)l\,dy\Big)^{\frac12}\times\Big({}^tlD(U,T)l\Big)^{\frac12}\\
&6e^{4\frac{\|T\|_\alpha}{R^\alpha}+\mathrm{Osc}(U)}(e^{8\frac{\|T\|_\alpha}{R^\alpha}}-1)^{\frac12}
\end{aligned}\tag{4.20}$$

This can be seen using (4.18), (4.11) and by Cauchy Schwarz inequality (the computation is similar to the one in (4.13)); Voigt-Reiss' inequality ($D(U)\geq e^{-2\,\mathrm{Osc}(U)}$) and noticing that for $y\in[0,1]^d$,$|T(x+y/R)-T(x)|\leq\|T\|_\alpha/R_\alpha$.
We now want to estimate I_3. Noting that

$$\nabla_y\Big(e^{-2(U(Rx+y)+T(x+\frac{y}{R}))}(l-\nabla_x\chi_l(x+\frac{y}{R},0))\Big)=0$$

$$\nabla\chi_.^U(Rx+y)=\nabla_y\chi_.^U(Rx+y)$$

and integrating by parts in y, one obtains

$$\begin{aligned}
I_3=&\sum_{i=1}^d\int_{x\in\mathbb{T}^d,y^i\in\partial^i([0,1]^d)}\Big(e^{-2T(x+\frac{y^i+e_i}{R})}(l-\nabla_x\chi_l(x+\frac{y^i+e_i}{R},0))\\
&-e^{-2T(x+\frac{y^i}{R})}(l-\nabla_x\chi_l(x+\frac{y^i}{R},0))\Big).e_i\chi_.^U(Rx+y^i)\\
&(\nabla\chi_l^{D(U),T}(x)-l)\frac{e^{-2U(Rx+y^i)}}{\int_{\mathbb{T}^d}e^{-2U(z)}dz\int_{\mathbb{T}^d}e^{-2T(z)}dz}dx\,dy^i
\end{aligned}$$

Where we have used the notation

$$\partial^i([0,1]^d)=\{x\in[0,1]^d\,:\,x_i=0\}$$

Let us introduce

$$I_4 = \sum_{i=1}^{d} \int_{x\in\mathbb{T}^d, y^i\in\partial^i([0,1]^d)} \Big(- \nabla_x \chi_l(x + \frac{y^i + e_i}{R}, 0) + \nabla_x \chi_l(x + \frac{y^i}{R}, 0)\Big).e_i$$
$$\chi^U_{\cdot}(Rx + y^i)(\nabla \chi_l^{D(U),T}(x) - l)e^{-2T(x+\frac{y^i}{R})} \frac{e^{-2U(Rx+y^i)}}{\int_{\mathbb{T}^d} e^{-2U(z)}dz \int_{\mathbb{T}^d} e^{-2T(z)}dz} dx\, dy^i \tag{4.21}$$

It is easy to obtain

$$|I_4 - I_3| \leq d e^{3\,\mathrm{Osc}(U)} e^{\frac{2\|T\|_\alpha}{R^\alpha}} (e^{\frac{2\|T\|_\alpha}{R^\alpha}} - 1)\Big({}^t l D(S_R U + T) l\Big)^{\frac{1}{2}} \|\chi^U_{\cdot}\|_\infty \Big({}^t l D(U,T) l\Big)^{\frac{1}{2}} \tag{4.22}$$

We will now put into evidence the fact that although $\chi_l(x, 0)$ is not periodic on $R^{-1}\mathbb{T}^d$ the distance (with respect to the natural H^1 norm) between the solution of the cell problem $\chi_l(x, 0)$ and its translation $\chi_l(x + e_k/R, 0)$ along the axis of the torus $R^{-1}\mathbb{T}^d$, is small. This is due to the presence of a fast period $R^{-1}\mathbb{T}^d$ in the decomposition $V = S_R U + T$. Using the standard property of the solution of the cell problem one obtains

$$\begin{aligned}
&\int_{\mathbb{T}^d} |\nabla \chi_l(x + \frac{e_k}{R}, 0) - \nabla \chi_l(x, 0)|^2 m_{S_R U+T}(dx) \\
&\quad = \int_{\mathbb{T}^d} |l - \nabla \chi_l(x, 0) + \nabla \chi_l(x + \frac{e_k}{R}, 0) - \nabla \chi_l(x, 0)|^2 m_{S_R U+T}(dx) - {}^t l D(S_R U + T) l \\
&\quad \leq e^{4\frac{\|T\|_\alpha}{R^\alpha}} \int_{\mathbb{T}^d} |l - \nabla \chi_l(x + \frac{e_k}{R}, 0))|^2 m_{\Theta_{\frac{e_k}{R}}(S_R U+T)}(dx) - {}^t l D(S_R U + T) l
\end{aligned} \tag{4.23}$$

which leads to

$$\int_{\mathbb{T}^d} |\nabla \chi_l(x + \frac{e_k}{R}, 0) - \nabla \chi_l(x, 0)|^2 m^{S_R U+T}(dx) \leq {}^t l D(S_R U + T) l (e^{4\frac{\|T\|_\alpha}{R^\alpha}} - 1) \tag{4.24}$$

Combining this inequality with the definition (4.21) of I_4, Cauchy-Schwarz inequality proves that

$$\begin{aligned}
|I_4| \leq & \sum_{i=1}^{d} \int_{y^i\in\partial^i([0,1]^d)} \Big(\int_{x\in\mathbb{T}^d} ((-\nabla_x \chi_l(x + \frac{y^i + e_i}{R}, 0) \\
& + \nabla_x \chi_l(x + \frac{y^i}{R}, 0)).e_i)^2 \frac{e^{-2U(Rx+y^i)-2T(x+\frac{y^i}{R})}}{\int_{\mathbb{T}^d} e^{-2U(z)}dz \int_{\mathbb{T}^d} e^{-2T(z)}dz} dx\Big)^{\frac{1}{2}} \\
& \Big(\int_{x\in\mathbb{T}^d} (\chi^U_{\cdot}(Rx + y^i)(\nabla \chi_l^{D(U),T}(x) - l))^2 \frac{e^{-2U(Rx+y^i)-2T(x+\frac{y^i}{R})}}{\int_{\mathbb{T}^d} e^{-2U(z)}dz \int_{\mathbb{T}^d} e^{-2T(z)}dz} dx\Big)^{\frac{1}{2}} dy^i
\end{aligned} \tag{4.25}$$

Combining this with (4.24) one obtains

$$|I_4| \leq (e^{\frac{8\|T\|_\alpha}{R^\alpha}} - 1)^{\frac{1}{2}} \Big({}^t l D(S_R U + T) l\Big)^{\frac{1}{2}} d \|\chi^U_{\cdot}\|_\infty e^{3\,\mathrm{Osc}(U)} \Big({}^t l D(U,T) l\Big)^{\frac{1}{2}} e^{\frac{2\|T\|_\alpha}{R^\alpha}} \tag{4.26}$$

Using lemma 4.16 to estimate $\|\chi^U_{\cdot}\|_\infty$ in (4.26) and combining (4.22) and (4.20) one obtains (4.15) and lemma 4.3, which proves the upper bound of theorem 3.2.

4.1.2.2 The purpose of this paragraph is to prove the estimate (4.16). First we will remind a theorem concerning elliptic equation with discontinuous coefficient from G. Stampacchia. Its proof in a more general form can be found in [44], chapter 5, theorem 5.4 (see also [43]).

Let us consider the operator (in the weak sense)$L = \nabla(A\nabla)$ defined on some open set $\Omega \subset \mathbb{R}^d$ (for $d \geq 3$) with smooth boundary $\partial\Omega$. A is a $d \times d$ matrix with bounded coefficients in $L^\infty(\Omega)$ such that for all $\xi \in \mathbb{R}^d$, $\lambda|\xi|^2 \leq {}^t\xi A\xi$ and for all i, j; $|A_{ij}| \leq M$, for some positive constant $0 < \lambda, M < \infty$.

Let $p > d \geq 3$. For $1 \leq i \leq d$ let $f_i \in L^p(\Omega)$

if $\chi \in H^1_{loc}(\Omega)$ is a local (weak) solution of the equation

$$\nabla\big(A\nabla\chi\big) = -\sum_{i=1}^{d} \partial_i f_i \tag{4.27}$$

then χ is in $L^\infty(\Omega)$ and if $x_0 \in \Omega$ and $R > 0$

Theorem 4.1. *The solution of (4.27) verify the following inequality (in the essential supremum sense with $\Omega(x_0, R) = \Omega \cap B(x_0, R)$)*

$$\max_{\Omega(x_0,\frac{R}{2})} |\chi| \leq K\Big[\{\frac{1}{R^d}\int_{\Omega(x_0,R)} \|\chi\|^2\}^{\frac{1}{2}} + \sum_{i=1}^{d} \|f_i\|_{L^p(\Omega(x_0,R))}\frac{R^{1-\frac{d}{p}}}{\lambda}\Big] \tag{4.28}$$

with $K = C_d(\frac{M}{\lambda})^{\frac{3d}{2}}$

The explicit dependence of the constants in M and λ have been obtained by following the proof of G. Stampacchia [44]. We will now prove (4.16) for $d \geq 3$ (For $d = 1$, this estimate is trivial, for $d = 2$, it is sufficient consider $U(x_1, x_2)$ as a function on T_1^3 to obtain the result). χ_l satisfies

$$\nabla\big(\exp(-2U)\nabla\chi_l\big) = l.\nabla\exp(-2U)$$

then by theorem 4.1 for $x_0 \in [0,1]^d$

$$\max_{B(x_0,\frac{1}{2})} |\chi_l| \leq C_d \exp(3\,\mathrm{Osc}(U)d)\big[\big(\int_{B(x_0,1)} |\chi_l|^2\big)^{\frac{1}{2}} + |l|\exp(2\,\mathrm{Osc}(U))\big]$$

Now by periodicity

$$\int_{B(x_0,1)} |\chi_l|^2 dx \leq \int_{\mathbb{T}^d} |\chi_l|^2 dx$$

and by Poincaré inequality (we assume $\int_{\mathbb{T}^d} \chi_l(x)dx = 0$)

$$\int_{\mathbb{T}^d} |\chi_l|^2 dx \leq C_d \int_{\mathbb{T}^d} |\nabla\chi_l|^2 dx$$

thus

$$\int_{B(x_0,1)} |\chi_l|^2 dx \leq C_d \exp(2\,\mathrm{Osc}(U)) \int_{\mathbb{T}^d} |\nabla\chi_l|^2 m_U(dx)$$

And since

$$\int_{\mathbb{T}^d} |l - \nabla\chi_l|^2 m_U(dx) = l^2 - \int_{\mathbb{T}^d} |\nabla\chi_l|^2 m_U(dx)$$

one has

$$\int_{\mathbb{T}^d} |\nabla\chi_l|^2 m_U(dx) \leq l^2$$

and the bound on $\|\chi_l\|_\infty$ is proven.

4.1.3 Quantitative multi-scale-homogenization: Lower bound in the theorem 3.2

4.1.3.1 As for the upper bound it is sufficient to prove theorem 3.2 assuming that U and T are smooth and we will use the notation introduced in the paragraph 4.1.2.1. We will prove below that

Lemma 4.5. *If $R \geq \|T\|_\alpha$ then for $\xi \in \mathbb{S}^{d-1}$*

$$\int_{\mathbb{T}^d} {}^t\xi D(S_R U + \Theta_y T)^{-1}\xi dy \leq (1 + C_{d,\mathrm{Osc}(U),\|U\|_\alpha,\alpha,\|T\|_\alpha} R^{-\alpha/2})^t\xi D(U,T)^{-1}\xi \tag{4.29}$$

This lemma with lemma 4.1 gives the lower bound in theorem 3.2. We now prove lemma 4.5. Let us introduce

$$P(x,y) = I_d - \frac{\exp(-2(S_R\Theta_y U + T))}{\int_{\mathbb{T}^d} \exp(-2(S_R\Theta_y U + T)(x))dx}(I_d - \nabla\chi(x,y)_.)D(S_R\Theta_y U + T)^{-1} \tag{4.30}$$

$$P^U(x) = I_d - \frac{\exp(-2U(x))}{\int_{\mathbb{T}^d} \exp(-2U(x))dx}(I_d - \nabla\chi^U(x)_.)D(U)^{-1} \tag{4.31}$$

and

$$P^{D(U),T}(x) = I_d - \frac{e^{-2T(x)}}{\int_{\mathbb{T}^d} e^{-2T(x)}dx} D(U)(I_d - \nabla\chi^{D(U),T}(x)_.)D(U,T)^{-1} \tag{4.32}$$

We remind that $P(x,y)$ minimize the well known variational formula associated to $D(S_R\Theta_y U + T)^{-1}$, that is why it will play for the lower bound in the theorem 3.2 the role played by the gradient of the solution of the solution of the cell problem $\nabla\chi(x,y)_.$ for the upper bound. More precisely, for $\xi \in \mathbb{S}^{d-1}$ one obtains as in the proof of the upper bound (by decomposing ξ here)

$$\begin{aligned}
&\int_{y\in\mathbb{T}^d} {}^t\xi D(S_R\Theta_y U + T)^{-1}\xi\, dy \\
&\quad= \int_{(x,y)\in(\mathbb{T}^d)^2} \Big(\int_{\mathbb{T}^d} e^{-2(S_R\Theta_y U+T)(z)}dz\Big) e^{2(S_R\Theta_y U+T)(x)t}\xi^t(I_d - P(x,y))\xi\, dx\, dy \\
&\quad\leq e^{\frac{2\|T\|_\alpha}{R^\alpha}}(I_1 + I_2)
\end{aligned} \tag{4.33}$$

with

$$\begin{aligned}
I_1 = &\int_{\mathbb{T}^d} e^{-2U(z)}\, dz \int_{\mathbb{T}^d} e^{-2T(z)}\, dz \int_{(x,y)\in(\mathbb{T}^d)^2} e^{2(S_R\Theta_y U+T)(x)t}\xi^t(I_d - P(x,y)) \\
&(I_d - P^U(Rx+y))(I_d - P^{D(U),T}(x))\xi\, dx\, dy
\end{aligned} \tag{4.34}$$

and

$$\begin{aligned}
I_2 = &\int_{\mathbb{T}^d} e^{-2U(z)}\, dz \int_{\mathbb{T}^d} e^{-2T(z)}\, dz \int_{(x,y)\in(\mathbb{T}^d)^2} e^{2(S_R\Theta_y U+T)(x)t}\xi^t(I_d - P(x,y)) \\
&P^U(Rx+y)(I_d - P^{D(U),T}(x))\xi\, dx\, dy
\end{aligned} \tag{4.35}$$

As for the upper bound, using Cauchy Schwarz inequality for the integration in x and y, and using

$$
\begin{aligned}
{}^t\xi D(U,T)^{-1}\xi = \int_{(x,y)\in(\mathbb{T}^d)^2} & e^{2(S_R\Theta_y U+T)(x)} \\
& \left((I_d - P^U(Rx+y))(I_d - P^{D(U),T}(x))\xi\right)^2 dx\,dy
\end{aligned}
\tag{4.36}
$$

one obtains that

$$
|I_1| \le e^{\frac{\|T\|_\alpha}{R^\alpha}} \Big(\int_{y\in\mathbb{T}^d} {}^t\xi D(S_R\Theta_y U+T)^{-1}\xi\, dy\Big)^{\frac12} \Big({}^t\xi D(U,T)^{-1}\xi\Big)^{\frac12}
\tag{4.37}
$$

Thus I_2 will be an error term and it will be proven below that

Lemma 4.6.

$$
\begin{aligned}
|I_2| \le & \Big(\int_{y\in\mathbb{T}^d} {}^t\xi D(S_R\Theta_y U+T)^{-1}\xi\, dy\Big)^{\frac12} \Big({}^t\xi D(U,T)^{-1}\xi\Big)^{\frac12} \\
& C_{d,\mathrm{Osc}(U),\|U\|_\alpha,\alpha} e^{4\frac{\|T\|_\alpha}{R^\alpha}} (e^{8\frac{\|T\|_\alpha}{R^\alpha}} - 1)^{\frac12}
\end{aligned}
\tag{4.38}
$$

Let us observe that combining the estimate (4.38) of lemma 4.6 with (4.37) and (4.33) proves lemma 4.5.
We will now prove the lemma 4.6. As it has been done in the proof of the upper bound it is easy to show that, with

$$
\begin{aligned}
I_3 = \int_{\mathbb{T}^d} e^{-2U(z)}\,dz \int_{\mathbb{T}^d} e^{-2T(z)}\,dz & \int_{(x,y)\in\mathbb{T}^d\times[0,1]^d} e^{2(U(Rx+y)+T(x+\frac{y}{R}))} \\
{}^t\xi^t(I_d - P(x+\frac{y}{R},0))P^U(Rx+y) & (I_d - P^{D(U),T}(x))\xi\, dx\,dy
\end{aligned}
\tag{4.39}
$$

one has

$$
|I_3 - I_2| \le 6e^{\mathrm{Osc}(U)} e^{\frac{4\|T\|_\alpha}{R^\alpha}} (e^{8\frac{\|T\|_\alpha}{R^\alpha}} - 1)^{\frac12} \Big({}^t\xi D(S_R U+T)\xi\Big)^{\frac12} \Big({}^t\xi D(U,T)\xi\Big)^{\frac12}
\tag{4.40}
$$

It will be proven in 4.1.3.2 that

Lemma 4.7. *There exists $d\times d\times d$ tensors H^U_{ijm} such that $H^U_{ijm} = -H^U_{jim} \in C^\infty(\mathbb{T}^d)$,*

$$
P^U_{im} = \sum_{j=1}^d \partial_j H^U_{ijm} \quad \textit{and} \quad \|H^U_{ijm}\|_\infty \le C_{d,\mathrm{Osc}(U),\|U\|_\alpha,\alpha}
\tag{4.41}
$$

Combining (4.41) with the explicit formula (4.30) for P one obtains

$$
\begin{aligned}
I_3 = & \frac{\int_{\mathbb{T}^d} e^{-2U(z)}\,dz \int_{\mathbb{T}^d} e^{-2T(z)}}{\int_{\mathbb{T}^d} \exp(-2(S_R U+T)(z))dz} \int_{(x,y)\in\mathbb{T}^d\times[0,1]^d} \sum_{i,j,k=1}^d ({}^t\xi^t(I_d - P^{D(U),T}(x)))_i \\
& \partial_k H^U_{i,k,j}(Rx+y)\Big((I_d - \nabla\chi_.(x+\frac{y}{R}))D(S_R U+T)^{-1}\xi\Big)_j\, dx\,dy
\end{aligned}
\tag{4.42}
$$

Thus, using the same notation as in the equation (4.21) and integrating by parts in y, one obtains

$$
\begin{aligned}
I_3 = & \frac{\int_{\mathbb{T}^d} e^{-2U(z)}\,dz \int_{\mathbb{T}^d} e^{-2T(z)}}{\int_{\mathbb{T}^d} \exp(-2(S_R U+T)(z))dz} \sum_{i,j,k=1}^d \int_{(x,y^k)\in\mathbb{T}^d\times\partial^k([0,1]^d)} ({}^t\xi^t(I_d - P^{D(U),T}(x)))_i \\
& H^U_{i,k,j}(Rx+y^k)\Big((\nabla\chi_.(x+\frac{y^k}{R}) - \nabla\chi_.(x+\frac{y^k+e_k}{R}))D(S_R U+T)^{-1}\xi\Big)_j\, dx\,dy^k
\end{aligned}
\tag{4.43}
$$

Which, using Cauchy Schwarz inequality, leads to

$$
\begin{aligned}
|I_3| \leq & C_d e^{2\,\mathrm{Osc}(U)} \sup_{ijk} \|H^U_{i,k,j}\|_\infty \sum_{k=1}^{d} \int_{y^k\in\partial^k([0,1]^d)} \Big(\int_{x\in\mathbb{T}^d} \big((I_d - P^{D(U),T}(x))\xi\big)^2 \\
& e^{2(U(Rx+y^k)+T(x+\frac{y^k}{R}))} dx \Big)^{\frac{1}{2}} \Big(\int_{x\in\mathbb{T}^d} \Big(((\nabla\chi_.(x+\frac{y^k}{R}) - \nabla\chi_.(x+\frac{y^k+e_k}{R})) \\
& D(S_R U + T)^{-1}\xi\Big)^2 e^{-2(U(Rx+y^k)+T(x+\frac{y^k}{R}))} dx\Big)^{\frac{1}{2}} dy^k
\end{aligned} \tag{4.44}
$$

Using bounds (4.41) and the equation (4.24) to control the natural H^1 distance between the solution of the cell problem $\chi_.(x+y^k/R)$ and its translation by e_k/R one obtains

$$
\begin{aligned}
|I_3| \leq & \Big(\xi D(S_R U + T)^{-1}\xi\Big)^{\frac{1}{2}} \Big({}^t\xi D(U,T)^{-1}\xi\Big)^{\frac{1}{2}} \\
& C_{d,\mathrm{Osc}(U),\|U\|_\alpha,\alpha} e^{4\frac{\|T\|_\alpha}{R^\alpha}} (e^{4\frac{\|T\|_\alpha}{R^\alpha}} - 1)^{\frac{1}{2}}
\end{aligned} \tag{4.45}
$$

Combining (4.45) and (4.40) one obtains (4.38), which proves lemma 4.38.

4.1.3.2 In this paragraph, we will prove lemma 4.7. Since for each $m \in \{1,\dots,d\}$, $P^U_{.,m}$ are divergence free vectors with mean 0 with respect to Lebesgue measure, by the proposition 4.1 of [28] there exists a skew-symmetric $\mathbb{T}^d$-periodic smooth matrices $H^U_{ij1},\dots,H^U_{ijd}$ ($H^U_{ijm} = -H^U_{jim}$) such that for all m

$$
P^U_{im} = \sum_{j=1}^{d} \partial_j H^U_{ijm} \tag{4.46}
$$

Moreover writing

$$
P^U_{.m} = \sum_{k\neq 0} p^k_{.m} e^{2i\pi(k.x)} \tag{4.47}
$$

the Fourier series expansion of P^U, one has (see the proposition 4.1 of [28])

$$
H^U_{njm} = \frac{1}{2i\pi} \sum_{k\neq 0} \frac{p^k_{nm}k_j - p^k_{jm}k_n}{k^2} e^{2i\pi(k.x)} \tag{4.48}
$$

Let us observe that

$$
H^U_{njm} = B^j_{nm} - B^n_{jm} \tag{4.49}
$$

where B^j_{nm} are the smooth $\mathbb{T}^d$-periodic solutions of $\Delta B^j_{nm} = \partial_j P^U_{nm}$. By theorem 4.1 (theorem 5.4, chapter 5 of [43]), if B_{nm} is chosen so that $\int_{\mathbb{T}^d} B_{nm}(x)dx = 0$ then $\|B^j_{nm}\|_\infty \leq C_d\|P^U_{nm}\|_\infty$. Now using theorem 1.1 of [32] it is easy to obtain that $\|\nabla\chi^U_l\|_\infty \leq C_{d,\mathrm{Osc}(U),\|U\|_\alpha,\alpha}$, combining this with (4.31), one obtains

$$
\|B^j_{nm}\|_\infty \leq C_{d,\mathrm{Osc}(U),\|U\|_\alpha,\alpha} \tag{4.50}
$$

Which leads to (4.41) by the equation (4.49).

4.1.4 Explicit formulas of effective diffusivities from level-3 large deviations. Proof of theorem 3.3

The equation (3.19) follows from the Voigt-Reiss inequality: for $U \in L^\infty(\mathbb{T}^d)$

$$D(U) \geq I_d\Big(\int_{\mathbb{T}^d} e^{2U(x)} \int_{\mathbb{T}^d} e^{-2U(x)}\Big)^{-1} \tag{4.51}$$

and the fact that, if $U \in C^\alpha(\mathbb{T}^d)$, then by the Varadhan's lemma and level-3 large deviation associated to the shift s_ρ,

$$\lim_{n\to\infty} \frac{1}{n} \ln\Big(\int_{\mathbb{T}^d} e^{\sum_{k=0}^{n-1} U(s_\rho^k x)} dx\Big) = P_\rho(U) \tag{4.52}$$

We refer to [39] for a more detailed proof of this statement.
In higher dimensions, when the medium is self-similar one can use the criterion (3.20) associated with the equation (3.19) to characterize ratios for which $D(V_0^n)$ does not converge to 0 with an exponential rate. The equation (3.17), i.e. the extension of the result (3.16) to dimension 2 is done by observing that

Proposition 4.1. *For $d = 2$ one has*

$$\begin{aligned}\lambda_{\max}\big(D(U)\big)\lambda_{\min}\big(D(-U)\big) &= \lambda_{\min}\big(D(U)\big)\lambda_{\max}\big(D(-U)\big)\\ &= \frac{1}{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx}\end{aligned} \tag{4.53}$$

From which one deduces that if $D(U) = D(-U)$ then

$$\lambda_{\min}(D(U))\lambda_{\max}(D(U)) = \Big(\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx\Big)^{-1} \tag{4.54}$$

Let us observe that the assumption $D(U) = D(-U)$ is satisfied if, for instance $-U_n(x) = U_n(-x)$ or $-U_n(x) = U_n(Ax)$ where A is an isometry of $\mathbb{R}^d$. And the existence of a reflection B such that $U(Bx) = U(x)$ ensures that $\lambda_{\min}(D(U)) = \lambda_{\max}(D(U))$. Thus these symmetry hypotheses combined with (3.17) ensure the validity of (3.18).
It would be interesting to extend the equation (3.17) of theorem 3.3 to more general cases and higher dimensions. Indeed the proposition 4.1 is deduced from the following proposition 4.2 that put into evidence a strong geometrical link between cohomology and homogenization.
Write $\mathcal{F}_{sol} = \Big\{p \in (C^\infty(T_1^d))^d |\, \mathrm{div}(p) = 0 \text{ and } \int_{T_1^d} p\,dx = 0\Big\}$ and $Q(U)$ the positive, definite, symmetric matrix associated to the following variational problem. For $l \in \mathbb{S}^d$

$${}^t l Q(U) l = \inf_{p\in\mathcal{F}_{sol}} \frac{\int_{T_1^d} |l-p|^2 \exp(2U)dx}{\int_{T_1^d} \exp(2U)dx} \tag{4.55}$$

Write in the increasing order $\lambda(D(U))_i$ and decreasing order $\lambda(Q(U))_i$ the eigenvalues of $D(U)$ and $Q(U)$.

Proposition 4.2. *For all $i \in \{1, \dots, d\}$*

$$\lambda(D(U))_i \lambda(Q(U))_i = \frac{1}{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx} \tag{4.56}$$

Now we will introduce a geometric interpretation of homogenization that will allow us to prove proposition 4.1 and equation (3.17) of theorem 3.3. Let $U \in C^\infty(\mathbb{T}^d)$. It is easy to obtain the following orthogonal decomposition

$$H = (L^2(m_U))^d = H_{pot} \oplus H_{sol} \tag{4.57}$$

Where H_{pot}, H_{sol} are the closure (with respect to the intrinsic norm $\|.\|_H$) of the sets of C_{pot}, C_{sol} the sets of smooth, $\mathbb{T}^d$-periodic, potential and solenoidal vector fields, i.e. with $\mathcal{C} = (C^\infty(\mathbb{T}^d))^d$

$$\mathcal{C}_{pot} = \Big\{\xi \in \mathcal{C} \,|\, \exists f \in C^\infty(\mathbb{T}^d) \text{ with } \xi = \nabla f\Big\} \tag{4.58}$$

$$\mathcal{C}_{sol} = \Big\{\xi \in \mathcal{C} | \exists p \in \mathcal{C} \text{ with } \operatorname{div}(p) = 0 \text{ and } \xi = p \exp(2U) \int_{\mathbb{T}^d} e^{-2U(x)} dx\Big\} \tag{4.59}$$

Thus H is a real Hilbert space equipped with the scalar product $(\xi, \nu)_H = \int_{\mathbb{T}^d} \xi(x).\nu(x) m_U(dx)$ and by the variational formulation (2.2), for $l \in \mathbb{R}^d$, ${}^t l D(U) l$ is the norm in H of the orthogonal projection of l on H_{sol} and $l = \nabla\chi_l + \exp(2U) p_l$ is the orthogonal decomposition of l.

$$\sqrt{{}^t l D(U) l} = \operatorname{dist}(l, H_{pot}) \tag{4.60}$$

Now by duality for all $\xi \in H$

$$\operatorname{dist}(\xi, H_{pot}) = \sup_{\delta \in \mathcal{C}_{sol}} \frac{(\delta, \xi)_H}{\|\delta\|_H} \tag{4.61}$$

From which we deduce the following variational formula for the effective diffusivity by choosing $\xi = l \in \mathbb{R}^d$

$${}^t l D(U) l = \sup_{p \in \mathcal{C} \ \operatorname{div}(p)=0} \frac{\big(\int_{\mathbb{T}^d} l.p dx\big)^2}{\int_{\mathbb{T}^d} p^2 \exp(2U) dx \int_{\mathbb{T}^d} \exp(-2U) dx} \tag{4.62}$$

Note that the equation (4.62) gives back Voigt-Reiss's inequality by choosing $p = l$.
Let $Q(U)$ be the positive, definite, symmetric matrix given by the variational formula (4.55). Then the following proposition is a direct consequence of the equation (4.62).

Proposition 4.3. *For all* $l \in \mathbb{S}^{d-1}$

$${}^t l D(U) l = \frac{1}{\int_{\mathbb{T}^d} \exp(2U) dx \int_{\mathbb{T}^d} \exp(-2U) dx} \sup_{\xi \in \mathbb{S}^{d-1}} \frac{(l.\xi)^2}{{}^t\xi Q(U) \xi} \tag{4.63}$$

Choosing an orthonormal basis diagonalizing $Q(U)$, it is an easy exercise to use this proposition in order to establish a one to one correspondence between the eigenvalues of $Q(U)$ and $D(U)$ to obtain the proposition 4.2.

4.1.4.1 Dimension two

In dimension two, the Poincaré duality establishes a simple correspondence between $Q(U)$ and $D(-U)$.

Proposition 4.4. *For* $d = 2$*, one has*

$$Q(U) = {}^t P D(-U) P \tag{4.64}$$

where P *stands for the rotation matrix*

$$P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.65}$$

Indeed by the Poincaré duality one has $\mathcal{F}_{sol} = \{P\nabla f \ : \ f \in C^\infty(\mathbb{T}^d)\}$ and the proposition 4.4 follows from the definition of $Q(U)$. The proposition 4.1 is then a direct consequence of the proposition 4.4 and one deduces from the equation (4.53) that if $D(U) = D(-U)$ then

$$\lambda_{\max}(D(U))\lambda_{\min}(D(U)) = \Big(\int_{\mathbb{T}^d} \exp(2U)dx \int_{\mathbb{T}^d} \exp(-2U)dx\Big) I_d \tag{4.66}$$

Which leads to the equation (3.17) of theorem 3.3 by theorem 3.1 of [39].

4.2 Sub-diffusive behavior from homogenization on infinitely many scales

4.2.1 Anomalous behavior of the exit times: Theorems 3.4 and 3.5.

4.2.1.1 In this subsection we will prove the asymptotic anomalous behavior of the mean exit times $\mathbb{E}_x[\tau(0,r)]$ defined as weak solutions of $L_V f = -1$ with Dirichlet conditions on $\partial B(0,r)$. Here $U_n \in C^1(\mathbb{T}^d)$, nevertheless we will assume first that those functions are smooth and prove quantitative anomalous estimates on $\mathbb{E}_x[\tau(0,r)]$ depending only on the values of $D(V_0^n)$, K_0 and K_α. Then, using standard estimates on the Green functions associated to divergence form elliptic operators (see for instance [44]) it is easy to check that the exit times $\mathbb{E}_x[\tau(0,r)]$ are continuous with respect to a perturbation of V in $L^\infty(B(0,r))$-norm. Using the density of smooth functions on $\overline{B(0,r)}$ in the set of bounded functions, we will then deduce that our estimates are valid for $U_n \in C^\alpha(\mathbb{T}^d)$.
Thus we can see the exit times as those associated to the solution of (3.23) and take advantage of the Ito formula.
The central lemma of the proof is lemma 3.1, which will be proven in the paragraph 4.2.1.2.
Writing $m_U^r(dx) = e^{-2U(x)}\,dx(\int_{B(0,r)} e^{-2U(x)}\,dx)^{-1}$, we will prove in the paragraph 4.2.1.3 that for $P \in C^\infty(\overline{B(0,r)})$

$$\begin{aligned}\int_{B(0,r)} \mathbb{E}_x^{U+P}\big[\tau(0,r)\big] m_{U+P}^r(dx) &\le e^{2\operatorname{Osc}_{B(0,r)}(P)} \int_{B(0,r)} \mathbb{E}_x^{U}\big[\tau(0,r)\big] m_{U+P}^r(dx)\\ &\ge e^{-2\operatorname{Osc}_{B(0,r)}(P)} \int_{B(0,r)} \mathbb{E}_x^{U}\big[\tau(B(0,r))\big] m_{U+P}^r(dx)\end{aligned} \tag{4.67}$$

We give here the outline of the proof (see [39] for $d = 1$). A perpetual homogenization process takes place over the infinite number of scales $0, \dots, n, \dots$ and the idea is still to distinguish, when one tries to estimate (3.29), the smaller scales which have already been homogenized ($0, \dots, n_{ef}$ called effective scales), the bigger scales which have not had a visible influence on the diffusion ($n_{dri}, \dots, \infty$ called drift scales because they will be replaced by a constant drift in the proof) and some intermediate scales that manifest the particular geometric structure of their associated potentials in the behavior of the diffusion ($n_{ef}+1, \dots, n_{dri}-1 = n_{ef}+n_{per}$ called perturbation scales because they will enter in the proof as a perturbation of the homogenization process over the smaller scales).
We will now use (3.25) and (4.67) to prove theorem 3.4. For that purpose, we will first fix the number of scales that one can consider as *homogenized* (we write ef for *effective*)

$$n_{ef}(r) = \sup\{n \ge 0 \ : \ e^{(n+1)(9d+15)K_0} R_n^2 \le C_1/(8C_d)r^2\} < \infty \tag{4.68}$$

where C^1 and C_d are the constants appearing in the left term of (3.25), next we fix the number of scales that will enter in the computation as a perturbation of the homogenization process (we write *per* for *perturbation*)

$$n_{per}(r) = \inf\{n \geq 0 \ : \ R_{n+1} \geq r\} - n_{ef}(r) \tag{4.69}$$

For $r > C_{d,K_0,\rho_{\max}}$, $n_{ef}(r)$ and $n_{per}(r)$ are well defined. Let us choose $U = V_0^{n_{ef}(r)}$, $P = V^{\infty}_{n_{ef}(r)+1}$ in (4.67), we will bound from above, $\mathrm{Osc}_{B(0,r)}(V^{\infty}_{n_{ef}(r)+1})$ by

$$\mathrm{Osc}(V^{n_{ef}(r)+n_{per}(r)}_{n_{ef}(r)+1}) + \|V^{\infty}_{n_{ef}(r)+n_{per}(r)+1}\|_{\alpha} r^{\alpha}$$

In the lower bound of (4.67) when $x \in B(0,r/2)$ we will bound $\mathbb{E}^U_x[\tau(0,r)]$ from below by $\mathbb{E}^U_x[\tau(x,r/2)]$ and in the upper bound when $x \in B(0,r)$ we will bound it from above by $\mathbb{E}^U_x[\tau(x,2r)]$. Then using (3.25) to control those exit times one obtains

$$\begin{aligned} \int_{B(0,r)} \mathbb{E}^V_x\left[\tau(B(0,r))\right] m_V^{B(0,r)}(dx) &\leq C_d e^{C_{K_\alpha,\alpha}+8n_{per}(r)K_0} \frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)} \\ &\geq C_d e^{-C_{K_\alpha,\alpha}-8n_{per}(r)K_0} \frac{r^2}{\lambda_{\max}\big(D(V^{0,n_{ef}(r)})\big)} \end{aligned} \tag{4.70}$$

Theorem 3.4 follows directly from the last inequalities by using the estimates (3.9) on $D(V_0^n)$, (2.11) on R_n and observing that

$$n_{per}(r) \leq \inf\{m \geq 0 \ : \ \frac{R_{m+n_{ef}(r)+1}}{R_{n_{ef}(r)+1}} \geq C_d e^{(n_{ef}(r)+2)(9d+15)K_0/2}\} \tag{4.71}$$

The proof of theorem 3.5 follows similar lines, the stability result (4.67) being replaced by the stability condition 3.1.

4.2.1.2 It is sufficient to prove the equation (3.25) for $x = 0$.
Let for $l \in \mathbb{S}^{d-1}$, χ_l be the $\mathbb{T}^d_R$-periodic solution of the cell problem associated to L_U with $\chi_l(0) = 0$.
Write ϕ_l the $\mathbb{T}^d_R$-periodic solution of the ergodicity problem $L_U\phi_l = |l - \nabla\chi_l|^2 - {}^t l D(U) l$ with $\phi_l(0) = 0$. Write $F_l(x) = l.x - \chi_l(x)$ and $\psi_l(x) = F_l^2(x) - \phi_l(x)$, observe that since $L_U F_l^2 = |l - \nabla\chi_l|^2$ it follows that

$$L_U \psi_l = {}^t l D(U) l \tag{4.72}$$

The following inequality will be used to show that $\sum_{i=1}^d \psi_{e_i}$ behaves like $|x|^2$

$$C_1|x|^2 - C_2(\|\chi_.\|^2_\infty + \|\phi_.\|_\infty) \leq \sum_{i=1}^d \psi_{e_i}(x) \leq C_3(|x|^2 + \|\chi_.\|^2_\infty + \|\phi_.\|_\infty) \tag{4.73}$$

Using theorem 4.1 (theorem 5.4, chapter 5 of [43]) to control F_l and ψ_l over one period (observing that $L_U F_l = 0$, $L_U\psi_l = -1$) and using $\chi_l = l.x - F_l$ and $\phi_l = F_l^2 - \psi_l$ one obtains easily that $\|\phi_.\|_\infty \leq C_d e^{(9d+13)\,\mathrm{Osc}(U)} R^2$, combining this estimate with (4.16) one obtains

$$\|\chi_.\|^2_\infty + \|\phi_.\|_\infty \leq C_d e^{(9d+13)\,\mathrm{Osc}(U)} R^2 \tag{4.74}$$

Since V has been assumed to be smooth (in a first step), we can use Ito formula to obtain

$$\psi_l(y_t) = \int_0^t \nabla\psi_l(y_s) d\omega_s \ + \ {}^t l D(U) l \, t \tag{4.75}$$

Now (e_i being an orthonormal basis of $\mathbb{R}^d$) write M_t the local martingale

$$M_t = \sum_{i=1}^{d} \psi_{e_i}(y_t) - \text{Trace}\big(D(U)\big)\, t \tag{4.76}$$

Define

$$\tau'(0,r) = \inf\{t \geq 0 \ : \ |\sum_{i=1}^{d} \psi_{e_i}(y_t)| = r\}$$

According to the inequality (4.73) one has

$$\tau'(0, C_1 r^2 - C_2(\|\chi.\|_\infty^2 + \|\phi.\|_\infty)) \leq \tau(0,r) \leq \tau'(0, C_3(r^2 + \|\chi.\|_\infty^2 + \|\phi.\|_\infty)) \tag{4.77}$$

Since $M_{t\wedge\tau'(0,r)}$ is uniformly integrable (easy to prove by using the inequalities (4.77)) one obtains

$$\mathbb{E}[\tau'(0,r)] = \frac{r}{\text{Trace}\big(D(U)\big)} \tag{4.78}$$

Thus, by using the inequality (4.74) and the Voigt-Reiss' inequality $D(U) \geq e^{-2\,\text{Osc}(U)}$ one obtains the equation (3.25).

4.2.1.3 The proof of the weak stability result (4.67) is based on the following obvious lemma that describes the monotony of Green functions as quadratic forms, i.e.

Lemma 4.8. *Let Ω be a smooth bounded open subset of $\mathbb{R}^d$. Assume that M, Q are symmetric smooth coercive matrices on $\overline{\Omega}$. Assume $M \leq \lambda Q$ with $\lambda > 0$, then for all $f \in C^0(\Omega)$, writing G_Q the Green functions of $-\nabla Q \nabla$ with Dirichlet condition on $\partial\Omega$*

$$\int_\Omega G_Q(x,y) f(y) f(x)\, dx\, dy \leq \lambda \int_\Omega G_M(x,y) f(y) f(x)\, dx\, dy \tag{4.79}$$

Proof. Let $f \in C^0(\overline{\Omega})$. Write ψ_M , ψ_Q the solutions of $-\nabla M \nabla \psi_M = f$ and $-\nabla Q \nabla \psi_Q = f$ with Dirichlet conditions on $\partial\Omega$. Observe that ψ_M and ψ_Q are the unique minimizers of $I_M(h,f)$ and $I_Q(h,f)$ with

$$I_M(h,f) = \frac{1}{2} \int_\Omega {}^t\nabla h M \nabla h dx - \int_\Omega h(x) f(x)\, dx \tag{4.80}$$

and $I_M(\psi_M, f) = -\frac{1}{2}\int_\Omega \psi_M(x) f(x)\, dx$. Observe that since $M \leq \lambda Q$

$$I_M(h,f) \leq \lambda I_Q(h, \frac{f}{\lambda}) \tag{4.81}$$

and the minimum of the right member in the equation (4.81) is reached at ψ_Q/λ. It follows that $\int_\Omega \psi_Q(x) f(x) \leq \lambda \int_\Omega \psi_M(x) f(x)$, which proves the lemma. ☐

Then, the equation (4.67) follows directly from this lemma by choosing $Q = e^{-2(U+P)}$, $M = e^{-2U}$ and observing that $\mathbb{E}_x^U[\tau(0,r)] = 2\int_{B(0,r)} G_{e^{-2U} I_d}(x,y) e^{-2U(y)} dy$.

4.2.2 Anomalous heat kernel tail: theorem 3.6

4.2.2.1 From the pointwise anomaly of the hitting times of theorem 3.5 one can deduce the anomalous heat kernel tail by adapting a strategy used by M.T. Barlow and R. Bass for the Sierpinski Carpet. This strategy is described in details in the proof of theorem 3.11 of [8] and we will give only the main lines of its adaptation.
We will estimate $\mathbb{P}_x[\tau(x,r) < t]$ and use $\mathbb{P}_x[|y_t| > r] \leq \mathbb{P}_x[\tau(x,r) < t]$ to obtain theorem 3.6.
Using the notations introduced in theorem 3.5 and $M := (d, K_0, K_\alpha, \alpha, \mu, \lambda_{\max})$, it will be shown in paragraph 4.2.2.2 that for $r > C(M, \rho_{\max})$ one has

$$\mathbb{P}_x[\tau(x,r) \leq t] \leq \frac{t}{r^{2+\sigma(r)(1+\gamma)} C_{19}(M)} + 1 - C_{20}(M) r^{-2\gamma\sigma(r)} \tag{4.82}$$

Now we will use lemma 3.14 of [8] given below (this is also lemma 1.1 of [9]).

Lemma 4.9. *Let $\xi_1, \xi_2, \dots, \xi_n$, V be non-negative r.v. such that $V \geq \sum_{i=1}^n \xi_i$. Suppose that for some $p \in (0,1)$, $a > 0$ and $t > 0$*

$$\mathbb{P}\big(\xi_i \leq t | \sigma(\xi_1, \dots, \xi_{i-1})\big) \leq p + at$$

Then

$$\ln \mathbb{P}(V \leq t) \leq 2\big(\frac{ant}{p}\big)^{\frac{1}{2}} - n \ln \frac{1}{p}$$

Let $n \geq 1$ and $g = \frac{r}{n}$. Define the stopping times S_i $i \geq 0$ by $S_0 = 0$ and

$$S_{i+1} = \inf\{t \geq S_i \ : \ |y_t - y_{S_i}| \geq g\}$$

Write $\xi_i = S_i - S_{i-1}$ for $i \geq 1$ Let $\mathcal{F}_t$ be the filtration of y_t and let $\mathcal{G}_i = \mathcal{F}_{S_i}$ Then it follows from the equation (4.82) that for

$$r/n > C(M, \rho_{\max}) \tag{4.83}$$

$$\begin{aligned}\mathbb{P}_x[\xi_{i+1} \leq t | (G)_i] &= \mathbb{P}_{y_{S_i}}[\tau(y_{S_i}, g) \leq t] \\ &\leq C_{21}(M) \frac{t}{g^{2+\sigma(r)(1+\gamma)}} + 1 - C_{20}(M) g^{-2\sigma(r)\gamma}\end{aligned}$$

Since $|y_{S_i} - y_{S_{i+1}}| = g$ it follows that $\mathbb{P}_x$ a.s. $|x - y_{S_n}| \leq r$. Thus

$$S_n = \sum_{i=1}^n \xi_i \leq \tau(x,r)$$

And by lemma 4.9 with

$$a = C_{21}(M) (\frac{n}{r})^{2+\sigma(r)(1+\gamma)} \quad p = 1 - C_{20}(M) (\frac{n}{r})^{2\sigma(r)\gamma}$$

One obtains

$$\ln \mathbb{P}_x[\tau(x,r) \leq t] \leq 2 \Big(\frac{n \, t \, C_{21} (\frac{n}{r})^{2+\sigma(r)(1+\gamma)}}{1 - C_{20}(\frac{n}{r})^{2\sigma(r)\gamma}} \Big)^{\frac{1}{2}} - n \ln \frac{1}{1 - C_{20}(\frac{n}{r})^{2\sigma(r)\gamma}} \tag{4.84}$$

Minimizing the right term in (4.84) over n under the constraint (4.83) and the assumptions (3.37), $\rho_{\min} > C_{6,M}$, one obtains theorem 3.6.

4.2.2.2 The equation (4.82) is an adaptation of lemma 3.16 of [8]. Observe that

$$\begin{aligned}\mathbb{E}_x[\tau(x,r)] \leq & t+\mathbb{E}_x[1(\tau(x,r)>t)\mathbb{E}_{y_t}[\tau(x,r)-t]] \\ \leq & t+\mathbb{P}_x[1(\tau(x,r)>t)] \sup_{y\in B(x,r)} \mathbb{E}_y[\tau(x,r)]\end{aligned}$$

Using $\forall y \in B(x,r)$, $\mathbb{P}_y$ a.s. $\tau(x,r) \leq \tau(y,2r)$ it follows by theorem 3.5 for $r > C(M,\rho_{\max})$

$$C_{33}(M)r^{2+\sigma(r)(1-\gamma)} \leq \mathbb{E}_x[\tau(x,r)] \leq t+\mathbb{P}_x[\tau(x,r)>t]C_{34}(M)r^{2+\sigma(r)(1+\gamma)}$$

Which leads to (4.82).

Acknowledgments The authors would like to thank Alain-Sol Sznitman, Stefano Olla and Alano Ancona [3] for stimulating discussions.

References

[1] *AIP Conference Proceedings. Electrical Transport and Optical Properties of Inhomogeneous Media*, number 40, 1977.

[2] G. Allaire and M. Briane. Multiscale convergence and reiterated homogenization. *Proc. Roy. Soc. Edinburgh Sect. A*, 126(2):297–342, 1996.

[3] Alano Ancona. Some results and examples about the behavior of harmonic functions and Green's functions with respect to second order elliptic operators. *preprint*, 1999.

[4] M. Avellaneda. Iterated homogenization, differential effective medium theory and applications. *Comm. Pure Appl. Matth.*, XL:527–554, 1987.

[5] M. Avellaneda. Homogenization and renormalization, the mathematics of multiscale random media and turbulent diffusion. In *Lectures in Applied Mathematics*, volume 31, pages 251–268, 1996.

[6] M. Avellaneda and A. Majda. Homogenization and renormalization of multiple-scattering expansions for green functions in turbulent transport. In *Composite Media and Homogenization Theory*, volume 5 of *Progress in Nonlinear Differential Equations and Their Applications*, pages 13–35, 1987.

[7] M. Avellaneda and A. Majda. Mathematical models with exact renormalization for turbulent transport. *Comm. Math. Phys.*, 131:381–429, 1990.

[8] M.T. Barlow. Diffusions on fractals. In *Lectures on probability theory and statistics*, pages 1–121. Saint-Flour, 1995, Springer, 1998.

[9] M.T. Barlow and R.F. Bass. Local times for brownian motion on the Sierpinski carpet. *Probab. Theory Relat. Fields*, 85:91–104, 1990.

[10] M.T. Barlow and R.F. Bass. Brownian motion and harmonic analysis on Sierpinski carpets. *Canadian J. Math*, 54:673–744, 1999.

[11] Gérard Ben Arous and Houman Owhadi. Super-diffusivity in a shear flow model from perpetual homogenization. *Submitted*, 2001. preprint available at http://dmawww.epfl.ch/∼owhadi/.

[12] A. Bensoussan, J. L. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structure*. North Holland, Amsterdam, 1978.

[13] Rabi Bhattacharya. Multiscale diffusion processes with periodic coefficients and an application to solute transport in porous media. *The Annals of Applied Probability*, 9(4):951–1020, 1999.

[14] Rabi Bhattacharya, Manfred Denker, and Alok Goswami. Speed of convergence to equilibrium and to normality for diffusions with multiple periodic scales. *Stochastic Processes and their Applications*, 80:55–86, 1999.

[15] S. Boucher. Modules effectifs de materiaux composites quasi homogenes et quasi isotropes, constitues d'une matrice elastique et d'inclusions elastiques, ii cas des concentrations finies en inclusions. *Rev. Metall.*, 22:31–36, 1997.

[16] D. A. G. Bruggerman. Berechnung verschiedener physkalisher konstanten von heterogenen substanzen. *Ann. Physik*, 24:634, 1935.

[17] M. P. Clearly, I. W. Chen, and S. M. Lee. Self-consistent techniques for heterogeneous media. *Am. Soc. Civil Eng. J. Eng. Mech.*, 106:861–887, 1980.

[18] A. Fannjiang. Anomalous diffusion in random flows. In *Mathematics of Multiscale Materials- The IMA volumes in mathematics and its apllications*, volume 99, pages 81–99, 1999.

[19] A. Fannjiang and T. Komorowski. Fractional brownian motion limit for motions in turbulence. *Ann. of Appl. Prob.*, 10(4), 2001.

[20] A. Fannjiang and G.C. Papanicolaou. Convection enhanced diffusion for periodic flows. *SIAM J. Appl. Math.*, 54:333–408, 1994.

[21] A. Fannjiang and G.C. Papanicolaou. Diffusion in turbulence. *Probab. Theory Related Fields*, 105(3):279–334, 1996.

[22] F. Furtado, J. Glimm, B. Lindquist, and F. Pereira. Multiple length scale calculus of mixing lenght growth in tracer floods. In F. Kovarik, editor, *Proceeding of the emerging technologies conference, Houston TX*, pages 251–259. Institute for Improved Oil Recovery, 1990.

[23] F. Furtado, J. Glimm, B. Lindquist, F. Pereira, and Q. Zhang. Time dependent anomalous diffusion for flow in multi-fractal porous media. In T.M.M. Verheggan, editor, *Proceeding of the workshop on numerical methods for simulation of multiphase and complex flow*, pages 251–259. Springer Verlag, New York, 1991.

[24] J. Glimm, B. Lindquist, F. Pereira, and R. Peierls. The multi-fractal hypothesis and anomalous diffusion. *Mat. Apl. Comput.*, 11(2):189–207, 1992.

[25] J. Glimm and Q. Zhang. Inertial range scaling of laminar shear flow as a model of turbulent transport. *Commun. Math. Phys.*, 146:217–229, 1992.

[26] B.M. Hambly, T. Kumagai, S. Kusuoka, and X.Y. Zhou. Transition density estimates for diffusion processes on homogeneous random Sierpinski carpets. *preprint*, 1998.

[27] M.B. Isichenko and J. Kalda. Statistical topography. ii. two-dimensional transport of a passive scalar. *J. Nonlinear Sci.*, 1:375–396, 1991.

[28] V.V. Jikov and S.M. Kozlov. Multiscaled homogenization. In V. Berdichevsky, V. Jikov, and G. Papanicolaou, editors, *Homogenization: Serguei Kozlov memorial volume*, pages 35–64. World Scientific, 1999.

[29] G. Keller. *Equilibrium States in Ergodic Theory*, volume 42 of *London mathematical society student texts*. Cambridge University press, 1998.

[30] T. Komorowski and S. Olla. On homogenization of time dependent random flows. *To appear in Probability Theory and Related Fields*, 2000.

[31] S. Kozlov. Multiscaled approach in homogenization. In *Proceeding of the Second Workshop on Composite Media and Homogenization theory*, pages 217–229. ICTP, Triestre, Italy, September 20 - October 1, 1993, World Scientific, 1995.

[32] Yan Yan Li and Michael Vogelius. Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients. *Arch. Rational Mech. Anal.*, 153:91–151, 2000.

[33] A.J. Majda and P.R. Kramer. Simplified models for turbulent diffusion: Theory, numerical modelling, and physical phenomena. *Physics reports*, 314:237–574, 1999. available at http://www.elsevier.nl/locate/physrep.

[34] R. A. McLaughlin. A study of the differential scheme for composite materials. *Int. J. Eng. Sci.*, 15:237–244, 1997.

[35] A. N. Norris. A differential scheme for the effective moduli of composites. *Mechanics of Materials*, 4:1–16, 1985.

[36] S. Olla. *Homogenization of Diffusion Processes in Random Fields*. Ecole Polytechnique, 1994. Cours Ecole Polytechnique.

[37] H. Osada. Isoperimetric dimension and estimates of heat kernels of pre-Sierpinski carpets. *Probab. Theory Relat. Fields*, 86:469–490, 1990.

[38] H. Owhadi. *Anomalous diffusion and homogenization on an infinite number of scales*. PhD thesis, EPFL - Swiss Federal Institute of Technology, 2001. Thesis no 2340, january 2001. Available at http://dmawww.epfl.ch/∼owhadi/.

[39] Houman Owhadi. Anomalous slow diffusion from perpetual homogenization. *Submitted*, 2001. preprint available at http://dmawww.epfl.ch/∼owhadi/.

[40] Houman Owhadi. Richardson law linked with Kolmogorov law in turbulent diffusion. *in preparation*, 2001.

[41] David Ruelle. *Thermodynamic formalism : the mathematical structures of classical equilibrium statistical mechanics*. Mass. Addison-Wesley, 1978.

[42] P.N. Sen, C. Scala, and M.H. Cohen. *Geophysics*, 46:781, 1981.

[43] G. Stampacchia. Le problème de dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, 15(1):189–258, 1965.

[44] G. Stampacchia. *Equations elliptiques du second ordre à coefficients discontinus*. Les Presses de l'Université de Montréal, 1966.

[45] Q. Zhang. A multi-scale theory of the anomalous mixing length growth for tracer flow in heterogeneous porous media. *J. Stat. Phys.*, 505:485–501, 1992.